Analysis on the Approximate Solutions of Integral Equations and Eigenvalue Problems

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Thesis submitted in partial fulfilment

for the Award of Degree

Doctor of Philosophy

by

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Shashank Kumar Shukla

Contents

List of Notations

$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\|a\|$	Absolute value of $a \in \mathbb{K}$
$\overline{z}$	Complex conjugate of $z \in \mathbb{C}$
$\overline{S}$	Closure of the set S
$\sup S$	Supremum of a subset $S \subset \mathbb{R}$
$\inf S$	Infimum of a subset $S \subset \mathbb{R}$
$(\mathcal{X}, \\|\cdot\\|)$	Normed linear space with norm $\\|\cdot\\|$
$\dim(\mathcal{X})$	Dimension of the linear space $\mathcal{X}$
$\mathcal{I}$	Identity operator on a space $\mathcal{X}$
$\mathcal{K} : \mathcal{X} \to \mathcal{X}$	Linear operator acting on space $\mathcal{X}$
$\mathcal{R}(\mathcal{K})$	Range space of the operator $\mathcal{K}$
$\mathcal{N}(\mathcal{K})$	Null space of the operator $\mathcal{K}$
$\mathrm{rank}(\mathcal{K})$	Dimension of the range space of $\mathcal{K}$
$(\mathcal{I} - \mathcal{K})^{-1}$	Inverse of the operator $\mathcal{I} - \mathcal{K}$ (if it exists)
$\mathcal{K}x$	Image of x under the Fredholm integral operator $\mathcal{K}$
$\\|\mathcal{K}\\|$	Operator norm of $\mathcal{K}$
$\mathcal{K}(x)$	Image of x under the Urysohn integral operator $\mathcal{K}$
$\mathcal{X}_n$	Finite-dimensional approximating space
$\mathcal{Q}_n$	Interpolatory projection operator
$\mathcal{P}_n$	Orthogonal projection operator
π_n	Projection operator (orthogonal or interpolatory)
$x_n \to x$	Sequence $\{x_n\}$ converges to x
$\mathcal{B}(x, \delta_0)$	Closed ball centered at x with radius δ_0
$\mathcal{B}(0, 1)$	Closed unit ball

$\mathcal{BL}(\mathcal{X},\mathcal{Y})$	Space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$
$\mathcal{BL}(\mathcal{X})$	Space of bounded linear operators on $\mathcal{X}$
$\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_k$	$\{(x_1, x_2, \ldots, x_k) : x_j \in \mathcal{X}_j,\ j = 1, 2, \ldots, k\}$
$\mathcal{X}^k$	Cartesian product $\mathcal{X} \times \mathcal{X} \times \cdots \times \mathcal{X}$ (k times)
$\mathcal{BL}(\mathcal{X}^k,\mathcal{Y})$	Space of bounded k-linear operators from $\mathcal{X}^k$ to $\mathcal{Y}$
$x \circ y$	Composition of mappings x and y
$\sigma(\mathcal{K})$	Spectrum of the operator $\mathcal{K}$
$\rho(\mathcal{K})$	Resolvent set of the operator $\mathcal{K}$
x'	First derivative of the function x
x''	Second derivative of the function x
$x^{(i)}$	i-th derivative of the function x
$\|x\|_\infty$	Supremum norm, $\sup\{\lvert x(t)\rvert : t \in [0,1]\}$
$\|x\|_{\alpha,\infty}$	$\max_{0\le j\le\alpha} \left\|x^{(j)}\right\|_\infty$
$\int_0^1 x(t)\,dt$	Riemann integral of x over $[0,1]$
$\langle\cdot,\cdot\rangle$	$\langle x, y\rangle = \int_0^1 x(t)y(t)\,dt$
$\mathcal{L}^\infty[0,1]$	Space of equivalence classes of essentially bounded functions on $[0,1]$
$\mathcal{L}^2[0,1]$	Space of square-integrable Lebesgue measurable functions on $[0,1]$
$\mathcal{C}[0,1]$	Space of continuous real-valued functions on $[0,1]$ with the supremum norm
$\mathcal{C}^\alpha[0,1]$	Space of α-times continuously differentiable functions on $[0,1]$ with the supremum norm

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Abstract

Integral equations play a central role in applied mathematics, modeling numerous problems arising in physics and engineering. This research is devoted to the numerical approximation of integral equations of the second kind, where the integral operator $\mathcal{K}$ may be linear (Fredholm) or nonlinear (Urysohn), as well as to the associated eigenvalue problems for compact linear integral operators. Since analytical solutions are rarely available, the operator $\mathcal{K}$ is approximated by a finite-rank operator $\mathcal{K}_n$, reducing the integral equation to a finite system of linear or nonlinear equations and transforming the eigenvalue problem into a matrix eigenvalue problem. Such problems can be efficiently treated using computational methods. Two principal approaches for constructing $\mathcal{K}_n$ are quadrature-based approximations (Nyström methods) and projection-based methods. The latter include Galerkin, collocation, and their modified variants, whose convergence and error behaviour are analyzed using tools from functional analysis.

This thesis investigates projection-based numerical methods for integral equations of the second kind and the associated eigenvalue problems for compact integral operators. Both linear Fredholm integral equations and nonlinear Urysohn integral equations are considered. Finite-dimensional approximation spaces consisting of piecewise polynomials of even degree are employed, together with interpolatory projections based on equidistant collocation points (not necessarily Gauss points), to construct classical, iterated, and modified projection methods. The corresponding eigenvalue problem for compact linear operators is analyzed using both orthogonal and interpolatory projection methods. Convergence rates are established for smooth kernels as well as for Green's function type kernels lacks smoothness along the diagonal. The modified projection method is shown to yield significantly higher accuracy while preserving the size of the resulting algebraic system, with only a marginal increase in computational effort. The theoretical findings are validated through numerical experiments that confirm the predicted orders of convergence.

Chapter 1

Introduction

The study of integral equations plays a central role in both pure and applied mathematics, providing a versatile framework for modeling and analyzing a wide range of problems arising in physics, engineering, and other scientific disciplines. Such equations frequently appear in the mathematical formulation of phenomena including heat conduction, wave propagation, potential theory, elasticity, and fluid dynamics, often as reformulations of boundary value problems for differential equations. A substantial literature exists on both the theoretical foundations and practical applications of integral equations. Among the classical references in the literature, we particularly highlight the contributions of Atkinson [13], Hackbusch [35], Hochstadt [37], Kress [46], Mikhlin [61], Pogorzelski [69], Schmeidler [79], Widom [92], and Zabreyko et al. [94], along with the references cited therein.

Among the various classes of integral equations, Fredholm integral equations of the second kind have received particular attention due to their well-posedness, in the sense that they admit a unique solution which depends continuously on the given data, together with their favourable analytical properties and stable numerical behaviour. In addition to their role in solving boundary value problems, these equations naturally give rise to eigenvalue problems associated with compact integral operators, which are fundamental in stability analysis, resonance phenomena, and vibration problems. The accurate numerical approximation of eigenvalues and eigenfunctions is therefore of significant importance in many applications. Beyond linear models, several practical problems lead to nonlinear integral equations, most notably Urysohn and Hammerstein equations, which arise in areas such as nonlinear boundary value problems, population dynamics, fluid flow, and nonlinear potential theory.

In many practical situations, obtaining exact solutions to integral equations or their associated eigenvalue problems is extremely difficult or rarely attainable. Therefore, numerical methods play a crucial role by providing approximate solutions that can be computed efficiently and refined to the desired level of accuracy. A common numerical strategy is to replace the original infinite-dimensional problem by a finite-dimensional one that can be solved using computational tools such as MATLAB, Python, and related software. Projection-based methods are widely employed for this purpose, including Galerkin methods, collocation methods, and their iterated and modified variants. These approaches project the problem onto suitable finite-dimensional subspaces, leading to systems of algebraic equations while preserving essential properties of the original operator, including spectral features in the case of eigenvalue problems.

The effectiveness of projection methods strongly depends on the choice of basis functions and projection operators. Appropriate choices can significantly improve convergence rates and numerical accuracy, whereas unsuitable selections may result in slow convergence. Interpolatory projection methods are particularly appealing due to their simplicity and practical efficiency. The development and analysis of accurate and efficient numerical methods for integral equations and their eigenvalue problems remain active areas of research. In this thesis, we investigate projection-based numerical methods for Fredholm and Urysohn integral equations of the second kind, as well as the associated eigenvalue problems. Both theoretical convergence analysis and numerical experiments are presented to demonstrate the effectiveness and reliability of the proposed approaches.

1.1 General Introduction

Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$ denote the Banach space of all equivalence classes of essentially bounded measurable functions on $[0,1]$, equipped with the supremum norm. Let $\Omega \times \mathbb{R} = [0,1] \times [0,1] \times \mathbb{R}$, and let $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Denote by $\mathcal{C}[0,1]$ the Banach space of all real-valued continuous functions on $[0,1]$, endowed with the supremum norm. We define the nonlinear integral operator $\mathcal{K} : \mathcal{L}^\infty[0,1] \to \mathcal{C}[0,1]$ by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1], \quad x \in \mathcal{X}.$$

Such an operator is referred to as a Urysohn integral operator. Since the kernel κ is continuous, the operator $\mathcal{K}$ is compact. For a given function f, we consider the corresponding Urysohn integral equation of the second kind

$$x - \mathcal{K}(x) = f. \tag{1.1.1}$$

A widely studied subclass of (1.1.1) is the Hammerstein integral equation, which arises when the kernel admits the separable form $\kappa(s,t,u) = \kappa(s,t)g(t,u)$. In this case, the equation becomes

$$x(s) - \int_0^1 \kappa(s,t)\, g\big(t, x(t)\big)\, dt = f(s), \quad s \in [0,1].$$

Such equations frequently appear in the reformulation of boundary value problems for ordinary and partial differential equations, where $\kappa(s,t)$ typically represents a Green's function; see, for example, [45, 93].

Another important special case is the Fredholm integral equation of the second kind, obtained when the kernel κ is independent of the unknown function x, that is, $\kappa(s,t,x(t)) = \kappa(s,t)$. In this case, the equation reduces to

$$x(s) - \int_0^1 \kappa(s,t)\, x(t)\, dt = f(s), \quad s \in [0,1],$$

where $\kappa : \Omega \to \mathbb{R}$ is a given continuous kernel. The associated linear Fredholm integral operator is defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt, \quad s \in [0,1], \quad x \in \mathcal{X}. \tag{1.1.2}$$

In this setting, $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a compact linear operator, and one typically takes $\mathcal{X} = \mathcal{C}[0,1], \text{or}, \mathcal{L}^2[0,1]$, where $\mathcal{L}^2[0,1]$ denotes the space of all square-integrable real-valued functions on $[0,1]$. The Fredholm integral equation of the second kind can then be written in the form

$$x - \mathcal{K}x = f. \tag{1.1.3}$$

Equation (1.1.3) plays a fundamental role in the theory of integral equations and arises in numerous applications, including potential theory, scattering problems, and mathematical physics.

Closely related to (1.1.3) is the associated eigenvalue problem for the compact linear operator $\mathcal{K}$ given in (1.1.2), which consists of finding $0 \neq \lambda \in \mathbb{C}$ and a nontrivial function $\psi \in \mathcal{X}$ such that

$$\mathcal{K}\psi = \lambda\psi.$$

Eigenvalue problems of this form are of central importance in stability analysis, vibration problems, and spectral theory. The compactness of $\mathcal{K}$ guarantees that all nonzero eigenvalues are isolated and have finite multiplicity, making projection-based numerical methods particularly suitable for their approximation.

1.2 Preliminaries

Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and let $\mathcal{Y}$ be another Banach space. We consider operators $\mathcal{T}$ defined on subsets $\mathcal{D} \subseteq \mathcal{X}$ and taking values in $\mathcal{Y}$, that is, $\mathcal{T} : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$. Since our study concerns fixed-point equations in $\mathcal{X}$, the space $\mathcal{Y}$ will typically coincide with $\mathcal{X}$. For the image of $x \in \mathcal{X}$ under $\mathcal{T}$ we write $\mathcal{T}x$ if $\mathcal{T}$ is linear, and $\mathcal{T}(x)$ if it is nonlinear. If $\mathcal{T}$ is a linear operator, its operator norm is denoted by $\|\mathcal{T}\|$ and is defined by

$$\|\mathcal{T}\| = \sup\{\|\mathcal{T}x\| : x \in \mathcal{X},\ \|x\| = 1\}.$$

For a sequence of operators $\{\mathcal{T}_n\}$ defined on $\mathcal{D}$, we write $\mathcal{T}_n x \to \mathcal{T}x$ if

$$\|\mathcal{T}_n x - \mathcal{T}x\|_{\mathcal{Y}} \to 0 \quad \text{as } n \to \infty.$$

The sequence $\{\mathcal{T}_n\}$ is said to be pointwise convergent on $\mathcal{D}$ if this holds for every $x \in \mathcal{D}$. In the case where $\mathcal{T}_n$ and $\mathcal{T}$ are linear, we consider norm convergence as

$$\|\mathcal{T}_n - \mathcal{T}\| \to 0 \quad \text{as } n \to \infty,$$

which implies pointwise convergence on all of $\mathcal{X}$. For nonlinear operators, the appropriate analogue is uniform convergence on bounded sets.

The Fréchet derivative of an operator $\mathcal{T}$ at $x \in \mathcal{X}$ is the bounded linear operator $\mathcal{T}'(x) : \mathcal{X} \to \mathcal{Y}$ satisfying

$$\lim_{\|h\|_{\mathcal{X}} \to 0} \frac{\|\mathcal{T}(x+h) - \mathcal{T}(x) - \mathcal{T}'(x)h\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

If $\mathcal{T}$ is linear, then $\mathcal{T}'(x) = \mathcal{T}$ for all x, that is, Fréchet derivative of a linear operator is itself. An operator is continuous at x whenever it is Fréchet differentiable there. It is said to be continuously Fréchet differentiable at x if

$$\|\mathcal{T}'(y) - \mathcal{T}'(x)\| \to 0 \quad \text{as } y \to x.$$

Denote by $\mathcal{BL}(\mathcal{X})$ the space of bounded linear operators on $\mathcal{X}$. For $\mathcal{T} \in \mathcal{BL}(\mathcal{X})$, the resolvent set of $\mathcal{T}$ is defined by

$$\rho(\mathcal{T}) = \{z \in \mathbb{C} :\ (z\mathcal{I} - \mathcal{T})^{-1} \in \mathcal{BL}(\mathcal{X})\},$$

where $\mathcal{I}$ is the identity operator on $\mathcal{X}$. The spectrum of $\mathcal{T}$ is

$$\sigma(\mathcal{T}) = \mathbb{C} \setminus \rho(\mathcal{T}).$$

Let $z_1, z_2 \in \rho(\mathcal{T})$. Then the first resolvent identity is given as

$$(z_1\mathcal{I} - \mathcal{T})^{-1} - (z_2\mathcal{I} - \mathcal{T})^{-1} = (z_2 - z_1)(z_1\mathcal{I} - \mathcal{T})^{-1}(z_2\mathcal{I} - \mathcal{T})^{-1}.$$

Let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{BL}(\mathcal{X})$ and $z \in \rho(\mathcal{T}_1) \cap \rho(\mathcal{T}_2)$. Then the second resolvent identity is

$$(z\mathcal{I} - \mathcal{T}_1)^{-1} - (z\mathcal{I} - \mathcal{T}_2)^{-1} = (z\mathcal{I} - \mathcal{T}_1)^{-1}[\mathcal{T}_1 - \mathcal{T}_2](z\mathcal{I} - \mathcal{T}_2)^{-1}.$$

A set $\mathcal{D} \subseteq \mathcal{X}$ is said to be dense in $\mathcal{X}$ if every element of $\mathcal{X}$ can be approximated arbitrarily well by elements of $\mathcal{D}$; that is, for every $x \in \mathcal{X}$ and every $\varepsilon > 0$, there exists $m \in \mathcal{D}$ such that $\|x - m\| < \varepsilon$. Equivalently, $\mathcal{D}$ is dense in $\mathcal{X}$ if its closure satisfies $\overline{\mathcal{D}} = \mathcal{X}$.

A set $\mathcal{D} \subseteq \mathcal{X}$ is relatively compact if its closure is compact in $\mathcal{X}$.

A subset $\mathcal{D} \subseteq \mathcal{C}[0,1]$ is said to be equicontinuous at $s \in [0,1]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$t \in [0,1],\ |s - t| < \delta \quad \Rightarrow \quad |x(s) - x(t)| < \varepsilon, \quad \forall\, x \in \mathcal{D}.$$

A sequence of operators $\{\mathcal{T}_n\}$ defined on $\mathcal{D} \subseteq \mathcal{X}$ is called collectively compact if, for every bounded set $\mathcal{B} \subset \mathcal{D}$, the set $\{\mathcal{T}_n(\mathcal{B}) : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{X}$.

A family of operators $\mathcal{F}$ on $\mathcal{X}$ is called collectively compact if the set

$$\{\mathcal{T}(x) :\ \mathcal{T} \in \mathcal{F},\ x \in \mathcal{B}(0,1)\}$$

is relatively compact in $\mathcal{X}$, where $\mathcal{B}(0,1) = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is a closed unit ball in $\mathcal{X}$. Since every bounded subset $\mathcal{B}$ of $\mathcal{X}$ is contained in a scalar multiple of the closed unit ball $\mathcal{B}(0,1)$, relative compactness of $\{\mathcal{T}_n(\mathcal{B})\}$ for all bounded $\mathcal{B}$ is equivalent to relative compactness of $\{\mathcal{T}_n(x) : x \in \mathcal{B}(0,1)\}$. Hence the sequence version and the family version of collective compactness describe the same concept. In Banach spaces, relative compactness is equivalent to total boundedness or sequential compactness

The following fundamental results from functional analysis will be used in the subsequent analysis.

Uniform Boundedness Principle (Banach-Steinhaus Theorem). Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a normed linear space. Let $\{\mathcal{T}_\alpha\} \subset \mathcal{BL}(\mathcal{X}, \mathcal{Y})$ be a family of bounded linear operators such that

$$\sup_\alpha \|\mathcal{T}_\alpha x\|_{\mathcal{Y}} < \infty \quad \text{for every } x \in \mathcal{X}.$$

Then the family $\{\mathcal{T}_\alpha\}$ is uniformly bounded, i.e.,

$$\sup_{\alpha} \|\mathcal{T}_\alpha\| < \infty.$$

Arzelà-Ascoli Theorem. A subset $\mathcal{D} \subseteq \mathcal{C}[0,1]$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $[0,1]$, that is,

$$\sup_{x\in\mathcal{D}} \|x\|_\infty < \infty,$$

and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|s-t| < \delta \quad \Rightarrow \quad |x(s) - x(t)| < \varepsilon \quad \forall\, x \in \mathcal{D}.$$

Hahn-Banach Theorem. Let $\mathcal{X}$ be a normed linear space and let $\mathcal{M} \subseteq \mathcal{X}$ be a subspace. If $f : \mathcal{M} \to \mathbb{R}$ (or $\mathbb{C}$) is a bounded linear functional, then there exists an extension $F : \mathcal{X} \to \mathbb{R}$ (or $\mathbb{C}$) such that

$$F|_{\mathcal{M}} = f \quad \text{and} \quad \|F\| = \|f\|.$$

Approximating Subspaces and Projection Operators

Let $\{\mathcal{X}_n\}_{n\geq 1}$ be a sequence of finite-dimensional subspaces of the Banach space $\mathcal{X}$. We say that $\{\mathcal{X}_n\}$ is a sequence of approximating spaces if the union $\bigcup_{n\geq 1} \mathcal{X}_n$ is dense in $\mathcal{X}$, that is,

$$\overline{\bigcup_{n\geq 1} \mathcal{X}_n} = \mathcal{X}.$$

In our analysis, the spaces $\mathcal{X}_n$, consisting of piecewise polynomial functions of even degree defined on a uniform partition of $[0,1]$, form an approximating sequence since their union is dense in $\mathcal{C}[0,1]$; accordingly, all approximation and convergence results are understood with respect to the $\|\cdot\|_\infty$-norm for functions in $\mathcal{C}[0,1]$, while the spaces $\mathcal{X}_n$ are naturally viewed as subspaces of $\mathcal{L}^\infty[0,1]$.

Corresponding to each approximating space $\mathcal{X}_n$, we consider the projection operator $\pi_n : \mathcal{X} \to \mathcal{X}_n$, which is linear and satisfies

$$\pi_n x = x \quad \text{for all } x \in \mathcal{X}_n.$$

Since π_n is a projection, we have $\pi_n^2 = \pi_n$, and hence

$$\left\|\pi_n^2\right\| = \|\pi_n\| \quad \text{which implies} \quad \|\pi_n\| \geq 1.$$

Thus, associated with the sequence of approximating spaces $\{\mathcal{X}_n\}_{n\geq 1}$, we obtain a corresponding sequence of projections $\{\pi_n\}_{n\geq 1}$.

Since the union $\bigcup_{n\geq 1} \mathcal{X}_n$ is dense in $\mathcal{X}$, one expects that $\pi_n x \to x$ for every $x \in \mathcal{X}$. This convergence indeed holds for every x belonging to the dense subset $\bigcup_{n\geq 1} \mathcal{X}_n$. By the Principle of Uniform Boundedness, pointwise convergence on all of $\mathcal{X}$ is equivalent to the uniform boundedness of the operator norms $\{\|\pi_n\|\}$.

Proposition 1. *Let $\{\pi_n\}$ be a sequence of projections on a Banach space $\mathcal{X}$. Then the following statements are equivalent:*

1. *$\pi_n x \to x$ for every $x \in \mathcal{X}$;*

2. *$\sup\limits_n \|\pi_n\| < \infty$ (equivalently, there exists $p > 0$ such that $\|\pi_n\| \leq p$ for all n).*

This characterization provides the basic functional-analytic framework underlying the convergence analysis of the projection-based numerical methods developed in this thesis.

1.3 Linear Integral Equations

Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$ and set $\Omega = [0,1] \times [0,1]$. We consider the linear integral operator $\mathcal{K}$ defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where the kernel $\kappa : \Omega \to \mathbb{R}$ is continuous. Then the integral operator $\mathcal{K}$ maps each function in $\mathcal{L}^\infty[0,1]$ to a continuous function on $[0,1]$, and hence defines a bounded linear operator from $\mathcal{L}^\infty[0,1]$ into $\mathcal{C}[0,1]$. Moreover, the image of any bounded set under $\mathcal{K}$ is relatively compact by the Arzelà-Ascoli theorem. Therefore, $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a compact linear operator, see Atkinson [13]. We are interested in a computable approximate solution of the following Fredholm integral equation of the second kind

$$x - \mathcal{K}x = f, \tag{1.3.1}$$

where 1 is not an eigenvalue of $\mathcal{K}$. Since $(\mathcal{I} - \mathcal{K})^{-1}$ is a bounded linear operator, the above equation has a unique solution.

A standard technique to solve (1.3.1) approximately is to replace $\mathcal{K}$ by a finite-rank operator. The approximate solution is then obtained by essentially solving a system of linear equations. We describe below two types of approximation methods, namely, methods based on projections and a method based on a numerical quadrature formula.

1. Projection methods, and
2. Nyström methods.

The projection methods relies on a sequence of continuous finite-rank projection operators that converge pointwise to the identity operator, which are then used to construct approximations of the operator $\mathcal{K}$ The methods include the classical Galerkin method associated with a sequence of orthogonal projections, the collocation method associated with a sequence of interpolatory projections, as discussed in Atkinson [13]. Sloan [83] first introduced iteration techniques to improve the Galerkin/collocation approximation and given iterated versions of the approximate solution. A recent method, called the modified projection method, proposed by Kulkarni [48], performs better than the classical methods in some respects. For projection-based methods, one can see [17, 21, 22, 30, 51, 75, 77, 80, 85, 86, 87, 88]. In the Nyström method, the integral in the definition of $\mathcal{K}$ is replaced by a convergent numerical quadrature formula, see Atkinson [13]. The discrete versions of the collocation and iterated collocation methods were studied by Joe [38], while the discrete Galerkin and iterated Galerkin methods were analyzed in [39]. Further developments and discussions on discrete projection methods for Fredholm integral equations can be found in [7, 12, 29]. In addition, Kulkarni-Rakshit introduced discrete modified projection and iterated modified projection methods for Fredholm integral equations and established rigorous convergence and superconvergence results; see [70]. In this thesis, we restrict ourselves to the projection methods.

Let the uniform partition of $[0,1]$ into n equal parts,

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n,$$

with subintervals $\Delta_j = [\, t_{j-1},\, t_j\,], \quad j = 1, 2, \ldots, n$. For non-negative integer r, define the piecewise polynomial approximating space

$$\mathcal{X}_n = \Big\{ x \in \mathcal{L}^\infty[0,1] : x|_{\Delta_j} \text{ is a polynomial of even degree } \leq 2r, j = 1, 2, \ldots, n \Big\}$$

That is, $\mathcal{X}_n$ denotes the space of piecewise polynomials of even degree at most $2r$ with respect to the above uniform partition. Since on each subinterval Δ_j the restriction of a function in $\mathcal{X}_n$ is a polynomial of degree at most $2r$, it is uniquely determined by $2r + 1$ coefficients. As the partition consists of n subintervals and no continuity is imposed across the partition points, the degrees of freedom on distinct subintervals are independent. Therefore, the total number of degrees of freedom is $n(2r + 1)$, and hence

$\dim \mathcal{X}_n = n(2r+1)$. Let

$$h = t_j - t_{j-1} = \frac{1}{n}, \quad \text{for } j = 1, 2, \ldots, n.$$

Here, we do not impose continuity at the partition points. Consequently, the space $\mathcal{X}_n$ is a subspace of $\mathcal{L}^\infty[0,1]$. We consider two types of projection operators mapping into $\mathcal{X}_n$. The first projection, denoted by $\mathcal{P}_n$, is defined as the restriction to $\mathcal{L}^\infty[0,1]$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ onto $\mathcal{X}_n$. The second projection, denoted by $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ be the projection map interpolating at $2r+1$ points in each subinterval of the partition. The construction and properties of these projection operators are discussed in the following subsections.

1.3.1 Orthogonal Projection

Let $\mathcal{X} = \mathcal{L}^2[0,1]$ denote the space of real-valued, square-integrable functions defined on the interval $[0,1]$. The inner product in $\mathcal{L}^2[0,1]$, denoted by $\langle \cdot, \cdot \rangle$, is defined as

$$\langle x, y \rangle = \int_0^1 x(t)y(t)\, dt, \quad \forall x, y \in \mathcal{L}^2[0,1].$$

For $\eta = 0, 1, \ldots, 2r$, let L_η represent the Legendre polynomial of degree η defined on the interval $[-1,1]$. For each $j = 1, \ldots, n$ and $\eta = 0, 1, \ldots, 2r$, we define the function $\psi_{j,\eta}(t)$ as follows:

$$\psi_{j,\eta}(t) = \begin{cases} \sqrt{\frac{2}{h}} L_\eta \left(\frac{2t - t_j - t_{j-1}}{h} \right), & t \in (t_{j-1}, t_j], \\ 0, & \text{otherwise.} \end{cases}$$

At the point t_0, we set $\psi_{1,\eta}(t_0) = \sqrt{\frac{2}{h}} L_\eta(-1)$. The supremum norm of $\psi_{j,\eta}$ is given by

$$\|\psi_{j,\eta}\|_\infty = \max_{t \in [t_{j-1}, t_j]} |\psi_{j,\eta}(t)| = \sqrt{\frac{2}{h}} \|L_\eta\|_\infty.$$

For functions $f, g \in \mathcal{C}(\Delta_j)$, we define the inner product on the interval Δ_j as

$$\langle f, g \rangle_{\Delta_j} = \int_{t_{j-1}}^{t_j} f(t)g(t)\, dt.$$

This inner product is symmetric, i.e., $\langle f, g \rangle_{\Delta_j} = \langle g, f \rangle_{\Delta_j}$. The supremum norm of f on Δ_j is defined as

$$\|f\|_{\Delta_j,\infty} = \max_{t \in [t_{j-1}, t_j]} |f(t)|.$$

Using this, we can bound the inner product as

$$\langle f, g\rangle_{\Delta_j} \leq \|f\|_{\Delta_j,\infty}\|g\|_{\Delta_j,\infty} h.$$

It is easy to see that the set $\{\psi_{j,\eta} : j = 1, \ldots, n, \eta = 0, \ldots, 2r\}$ forms an orthonormal basis for the space $\mathcal{X}_n$. Let $\mathbb{P}_{2r,\Delta_j}$ denote the space of polynomials of degree at most $2r$ on Δ_j. The orthogonal projection $\mathcal{P}_{n,j} : \mathcal{C}(\Delta_j) \to \mathbb{P}_{2r,\Delta_j}$ is defined by

$$\mathcal{P}_{n,j}x = \sum_{\eta=0}^{2r} \langle x, \psi_{j,\eta}\rangle_{\Delta_j} \psi_{j,\eta}.$$

This projection satisfies the following properties:

$$\langle \mathcal{P}_{n,j}x, y\rangle_{\Delta_j} = \langle x, \mathcal{P}_{n,j}y\rangle_{\Delta_j}, \quad \mathcal{P}_{n,j}^2 = \mathcal{P}_{n,j}, \quad \text{and} \quad \mathcal{P}_{n,j}\mathcal{P}_{n,i} = 0 \quad \text{for } i \neq j.$$

Additionally, we have the bound as

$$\|\mathcal{P}_{n,j}x\|_{\Delta_j,\infty} \leq \sum_{\eta=0}^{2r} \langle x, \psi_{j,\eta}\rangle_{\Delta_j} \|\psi_{j,\eta}\|_{\Delta_j,\infty} \leq 2\left(\sum_{\eta=0}^{2r} \|L_\eta\|_\infty^2\right)\|x\|_\infty.$$

Thus, an orthogonal projection $\mathcal{P}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ is defined as

$$\mathcal{P}_n x = \sum_{j=1}^{n} \mathcal{P}_{n,j}x.$$

By applying the Hahn-Banach extension theorem, following the approach in Atkinson et al. [8], the projection $\mathcal{P}_n$ can be extended to the space $\mathcal{L}^\infty[0,1]$. This extended projection satisfies $\mathcal{P}_n^2 = \mathcal{P}_n$ and

$$\|\mathcal{P}_n\| \leq 2\sum_{\eta=0}^{2r} \|L_\eta\|_\infty^2 = C_1. \tag{1.3.2}$$

If $x \in \mathcal{C}^{2r+1}(\Delta_j)$, then by Taylor's theorem we may write

$$x(t) = \sum_{k=0}^{2r} \frac{(t - t_{j-1})^k}{k!} x^{(k)}(t_{j-1}) + R_j(t), \quad t \in \Delta_j,$$

where the remainder term is given by

$$R_j(t) = \frac{(t - t_{j-1})^{2r+1}}{(2r+1)!} x^{(2r+1)}(\xi_j), \quad \xi_j \in (t_{j-1}, t_j).$$

Since $(t - t_{j-1})^k$ is a polynomial of degree $k \leq 2r$, and $\mathcal{P}_{n,j}$ denotes the orthogonal projection onto the space of polynomials of degree at most $2r$ on Δ_j, we have

$$\mathcal{P}_{n,j}(t - t_{j-1})^k = (t - t_{j-1})^k, \quad k = 0, 1, \ldots, 2r.$$

It follows immediately that

$$(\mathcal{I} - \mathcal{P}_{n,j})(t - t_{j-1})^k = 0, \quad k = 0, 1, \ldots, 2r,$$

and therefore

$$(\mathcal{I} - \mathcal{P}_{n,j})\, x = (\mathcal{I} - \mathcal{P}_{n,j})\, R_j.$$

Taking the uniform norm on Δ_j gives

$$\|(\mathcal{I} - \mathcal{P}_{n,j})x\|_{\Delta_j,\infty} \le \|\mathcal{I} - \mathcal{P}_{n,j}\| \, \|R_j\|_{\Delta_j,\infty}.$$

Using the elementary estimate $\|\mathcal{I} - \mathcal{P}_{n,j}\| \le 1 + \|\mathcal{P}_{n,j}\|$ and the boundedness assumption $\|\mathcal{P}_{n,j}\| \le C_1$, we obtain

$$\|(\mathcal{I} - \mathcal{P}_{n,j})x\|_{\Delta_j,\infty} \le (1 + C_1)\, \|R_j\|_{\Delta_j,\infty}.$$

For $t \in \Delta_j$ we have $|t - t_{j-1}| \le h$, from which it follows that

$$\|R_j\|_{\Delta_j,\infty} \le \frac{h^{2r+1}}{(2r+1)!} \left\|x^{(2r+1)}\right\|_{\Delta_j,\infty}.$$

Combining the above bounds yields the local estimate

$$\|(\mathcal{I} - \mathcal{P}_{n,j})x\|_{\Delta_j,\infty} \le C_2 \left\|x^{(2r+1)}\right\|_{\Delta_j,\infty} h^{2r+1},$$

where

$$C_2 = \frac{1 + C_1}{(2r+1)!}$$

is a constant independent of the h.

Finally, if $x \in \mathcal{C}^{2r+1}[0, 1]$, then summing over all subintervals gives the global estimate

$$\|(\mathcal{I} - \mathcal{P}_n)x\|_\infty \le C_2 \left\|x^{(2r+1)}\right\|_\infty h^{2r+1}. \tag{1.3.3}$$

In fact, let $\alpha \ge 0$ be an integer and define $\beta = \min\{\alpha,\, 2r+1\}$. Then, for any $x \in \mathcal{C}^\alpha[0, 1]$, the orthogonal projection $\mathcal{P}_n$ satisfies

$$\|(\mathcal{I} - \mathcal{P}_n)x\|_\infty \le C_2 \left\|x^{(\beta)}\right\|_\infty h^\beta.$$

Although the projection $\mathcal{P}_n$ is constructed using the $\mathcal{L}^2$-inner product, the convergence and error analysis are carried out in the supremum (uniform) norm, since uniform estimates are essential for establishing convergence of the approximate solutions of the integral equation and the associated eigenvalue problem.

1.3.2 Interpolatory Projection

We recall the uniform partition of the interval $[0,1]$,

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n,$$

with subintervals $\Delta_j = [\,t_{j-1},\, t_j\,], \quad j = 1, 2, \ldots, n$. Define the piecewise polynomial finite-dimensional approximating space

$$\mathcal{X}_n \;=\; \Big\{ x \in \mathcal{L}^\infty[0,1] \colon x\big|_{\Delta_j} \text{ is a polynomial of even degree } \leq 2r \Big\}.$$

Since on each subinterval Δ_j we have $2r+1$ coefficients, $\dim \mathcal{X}_n \;=\; n\,(2r+1)$. Within each subinterval Δ_j, choose the $2r+1$ equally spaced interpolation points

$$\tau_j^i = t_{j-1} + \frac{ih}{2r} = t_{j-1} + h\zeta_i, \quad i = 0, 1, \ldots, 2r,$$

where

$$\zeta_i = \frac{i}{2r} \in [0,1], \quad r \geq 1.$$

In the special case $r = 0$, we define

$$\tau_j = \frac{t_{j-1} + t_j}{2}.$$

Define the local interpolant $\mathcal{Q}_{n,j}x$ on Δ_j by

$$\mathcal{Q}_{n,j}x\big(\tau_j^i\big) \;=\; x\big(\tau_j^i\big), \quad i = 0, 1, \ldots, 2r.$$

Combining the local interpolants on each subinterval defines the global interpolatory operator $\mathcal{Q}_n \colon \mathcal{C}[0,1] \to \mathcal{X}_n$. Standard approximation arguments imply that

$$\mathcal{Q}_n x \;\to\; x \quad \text{uniformly on } [0,1], \quad \forall\, x \in \mathcal{C}[0,1].$$

If $\mathcal{Q}_n x$ is known to be continuous for the class of functions under consideration, we choose $\mathcal{X}_n$ to be the space of continuous piecewise polynomials of even degree $\leq 2r$. In this case, $\mathcal{X}_n \subset \mathcal{C}[0,1]$, and the mapping $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ acts as a projection.

In situations where the interpolant $\mathcal{Q}_n x$ is not necessarily continuous, we instead take $\mathcal{X}_n$ to be the space of discontinuous piecewise polynomials of even degree $\leq 2r$ defined on the same partition. In this setting, $\mathcal{X}_n \subset \mathcal{L}^\infty[0,1]$, and we extend $\mathcal{Q}_n$ so that it becomes a projection $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$, which is the situation considered in this thesis; see Atkinson et al. [8]. Since $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ is linear and bounded, and $\mathcal{C}[0,1]$ is dense in $\mathcal{L}^\infty[0,1]$, the Hahn-Banach theorem ensures that each bounded linear functional involved in the definition of $\mathcal{Q}_n$ admits an extension to $\mathcal{L}^\infty[0,1]$ with the same norm. Consequently, $\mathcal{Q}_n$ possesses a unique bounded linear extension to all of $\mathcal{L}^\infty[0,1]$, and this extended operator coincides with the original interpolatory operator on $\mathcal{C}[0,1]$. Thus, $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ is a bounded linear projection onto $\mathcal{X}_n$.

For each subinterval Δ_j, we define

$$\Psi_j(t) \;=\; \prod_{i=0}^{2r}\big(t-\tau_j^i\big), \quad t \in \Delta_j.$$

If $x \in \mathcal{C}[0,1]$, then by the classical result of Conte-de Boor [24], the local interpolation error can be expressed in divided difference form as

$$x(t) - \mathcal{Q}_{n,j}x(t) \;=\; \Psi_j(t)\,\big[\,\tau_j^0,\,\ldots,\,\tau_j^{2r},\,t\,\big]\,x, \quad t \in \Delta_j, \tag{1.3.4}$$

where $\big[\,\tau_j^0,\ldots,\tau_j^{2r},t\,\big]\,x$ denotes the $(2r+1)$-th divided difference of x at the indicated points. Denote the first divided difference of $x \in \mathcal{C}[0,1]$ at τ_j^0 by $\big[\tau_j^0\big]\,x = x(\tau_j^0)$, and the $(2r+1)$-th divided difference of x at the points $\tau_j^0,\ldots,\tau_j^{2r}$ by

$$\big[\,\tau_j^0,\ldots,\tau_j^{2r}\,\big]\,x \;=\; \frac{\big[\tau_j^1,\ldots,\tau_j^{2r}\big]\,x \;-\; \big[\tau_j^0,\ldots,\tau_j^{2r-1}\big]\,x}{\tau_j^{2r}-\tau_j^0}.$$

If $x \in \mathcal{C}^{2r+1}[0,1]$, then the divided difference reduces to

$$\big[\,\tau_j^0,\,\ldots,\,\tau_j^{2r},\,t\,\big]\,x = \frac{x^{(2r+1)}(\gamma_j)}{(2r+1)!}, \quad \text{for some } \gamma_j \in (t_{j-1},t_j).$$

Next, we introduce

$$\Phi_{2r+1}(\tau) \;=\; \prod_{i=0}^{2r}(\tau-\zeta_i), \quad \tau \in [0,1].$$

For $t \in \Delta_j$, we can rewrite Ψ_j in terms of Φ_{2r+1} as

$$\begin{aligned}\Psi_j(t) &= \prod_{i=0}^{2r}\big(t-\tau_j^i\big)\\ &= \Big(\frac{t-t_{j-1}}{h}-\zeta_0\Big)\cdots\Big(\frac{t-t_{j-1}}{h}-\zeta_{2r}\Big)\,h^{2r+1}\\ &= \Phi_{2r+1}\left(\frac{t-t_{j-1}}{h}\right)\,h^{2r+1}.\end{aligned}$$

Combining (1.3.4) and the above estimate, we obtain the local error estimate

$$\sup_{t\in\Delta_j}\big|\,x(t)-(\mathcal{Q}_{n,j}x)(t)\,\big| \le \frac{1}{(2r+1)!}\left(\sup_{t\in\Delta_j}|x^{(2r+1)}(t)|\right)\left(\sup_{\tau\in[0,1]}|\Phi_{2r+1}(\tau)|\right)h^{2r+1}.$$

Hence, if $x \in \mathcal{C}^{2r+1}(\Delta_j)$, there exists a constant $C_3 > 0$, independent of h, such that

$$\| x - \mathcal{Q}_{n,j}x \|_{\Delta_j,\infty} \ \leq \ C_3 \left\| x^{(2r+1)} \right\|_{\Delta_j,\infty} h^{2r+1},$$

where

$$C_3 = \frac{\|\Phi_{2r+1}\|_\infty}{(2r+1)!}.$$

Consequently, for any $x \in \mathcal{C}^{2r+1}[0,1]$, the global interpolation error satisfies

$$\| x - \mathcal{Q}_n x \|_\infty \ \leq \ C_3 \left\| x^{(2r+1)} \right\|_\infty h^{2r+1}.$$

More generally, let $\alpha \geq 0$ be an integer and define $\beta = \min\{\alpha,\, 2r+1\}$. Then, for any $x \in \mathcal{C}^\alpha[0,1]$, the following estimate holds:

$$\|x - \mathcal{Q}_n x\|_\infty \leq C_3 \left\|x^{(\beta)}\right\|_\infty h^\beta. \tag{1.3.5}$$

Properties of the Interpolation Polynomial

Consider the polynomial $\Psi_j(t) = (t-\tau_j^0)(t-\tau_j^1)\cdots(t-\tau_j^{2r})$, where $\tau_j^i,\ i = 0,1,\ldots,2r$, are the $2r+1$ evenly spaced interpolation points in the subinterval $[t_{j-1}, t_j]$. Let $c = \dfrac{t_{j-1}+t_j}{2}$ denote the midpoint of this interval. The interpolation points are symmetric with respect to c, in the sense that $\tau_j^i + \tau_j^{2r-i} = 2c$. Using this symmetry, for any $u \in [0, c - t_{j-1}]$ we write

$$\Psi_j(c-u) = \prod_{i=0}^{2r}\big((c-u) - \tau_j^i\big) = \prod_{i=0}^{2r}\big((c-\tau_j^i) - u\big).$$

Replacing the index i by $2r-i$ and using $c - \tau_j^i = -(c - \tau_j^{2r-i})$, we obtain

$$\Psi_j(c-u) = (-1)^{2r+1}\prod_{i=0}^{2r}\big((c+u) - \tau_j^{2r-i}\big) = (-1)^{2r+1}\Psi_j(c+u).$$

Since $2r+1$ is odd, this yields

$$\Psi_j(c-u) = -\Psi_j(c+u),$$

showing that Ψ_j is antisymmetric about the midpoint c.

As a direct consequence of this antisymmetry, we have the following observation.

Lemma 1.3.1. *Let $\tau_j^i,\ i = 0,1,\ldots,2r$, be $2r+1$ evenly spaced interpolation points in the interval $[t_{j-1}, t_j]$. Then the integral of the interpolation polynomial Ψ_j over $[t_{j-1}, t_j]$ vanishes, that is,*

$$\int_{t_{j-1}}^{t_j} \Psi_j(s)\, ds = 0.$$

Proof. Let $c = \dfrac{t_{j-1}+t_j}{2}$ denote the midpoint of the interval $[t_{j-1}, t_j]$. Since the interpolation points are evenly spaced, the midpoint c coincides with one of the nodes, namely $\tau_j^r = c$. From the antisymmetry property established earlier, we have

$$\Psi_j(c+u) = -\Psi_j(c-u) \quad \text{for all } u \in [0, c-t_{j-1}].$$

We split the integral at the midpoint c and perform a change of variables:

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \Psi_j(s)\,ds &= \int_{t_{j-1}}^{c} \Psi_j(s)\,ds + \int_{c}^{t_j} \Psi_j(s)\,ds \\ &= \int_0^{c-t_{j-1}} \Psi_j(c-u)\,du + \int_0^{t_j-c} \Psi_j(c+u)\,du.\end{aligned}$$

Since $c - t_{j-1} = t_j - c$, and using the antisymmetry of Ψ_j, the second integral becomes

$$\int_0^{c-t_{j-1}} \Psi_j(c+u)\,du = -\int_0^{c-t_{j-1}} \Psi_j(c-u)\,du.$$

Hence, the two integrals cancel each other, yielding

$$\int_{t_{j-1}}^{t_j} \Psi_j(s)\,ds = 0.$$

This completes the proof. □

This property plays a fundamental role in the subsequent error analysis of the interpolatory projection. In the classical interpolatory projection setting, superconvergence is obtained when Gauss points (i.e., the zeros of the Legendre polynomial) are used as collocation points. The present work shows that a similar improvement can also be achieved using uniformly distributed collocation points, provided the polynomial degree is even. Indeed, for polynomial spaces of degree at most $2r$, there are $2r+1$ uniformly distributed interpolation nodes on each subinterval. Since the number of interpolation nodes is odd, the midpoint of each subinterval is itself an interpolation node, resulting in a symmetric node distribution. Thus, the nodal interpolation polynomial $\Psi_j(t)$ is antisymmetric about the midpoint and satisfies

$$\int_{t_{j-1}}^{t_j} \Psi_j(t)\,dt = 0.$$

This cancellation eliminates the leading-order interpolation error, yielding an additional factor of h in the error estimate and, consequently, the superconvergence of the iterated and modified projection methods.

If this symmetry is absent, the cancellation property is lost, and the iterated/modified methods no longer achieve an improved convergence rate over the corresponding standard collocation method. This is illustrated in Example 1 of Chapter 2, where the interpolation point is not located at the midpoint of each subinterval, and

$$\frac{\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty}{\|(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty} \not\to 0,$$

violating the necessary condition for superconvergence. Hence, iteration does not improve the convergence rate. Similar observations are made in Example 6 of Chapter 3.

Therefore, the restriction to even-degree polynomial spaces is essential, as it guarantees the symmetry and resulting cancellation property required to obtain the improved convergence rates established in this work.

1.3.3 Projection Methods

We now present several projection methods and their variants for obtaining approximate solutions of (1.3.1). For completeness, we also include the corresponding error estimates, as established in Atkinson [13, Chapter 3] and Kulkarni [48]. These methods are developed in the context of Fredholm integral equations of the second kind and are based on finite-dimensional subspace approximations of the underlying operator. The analysis focuses on the convergence and accuracy of the resulting approximate solutions under suitable regularity assumptions on the kernel and the right-hand side function.

1. **Galerkin and Collocation Methods.**

   In the Galerkin method, the equation (1.3.1) is approximated by

   $$\varphi_n^G - \mathcal{P}_n\mathcal{K}\varphi_n^G = \mathcal{P}_n f,$$

   whereas in the collocation method, it is approximated by

   $$\varphi_n^C - \mathcal{Q}_n\mathcal{K}\varphi_n^C = \mathcal{Q}_n f.$$

   Let π_n denote either $\mathcal{P}_n$ or $\mathcal{Q}_n$, and let φ_n denote either φ_n^G or φ_n^C. Since $\mathcal{K}$ is compact and π_n converges pointwise to the identity operator, it follows that $\|\pi_n\mathcal{K}-\mathcal{K}\| \to 0$ as $n \to \infty$. Further, since $\mathcal{I}-\mathcal{K}$ is invertible, it follows that $\mathcal{I}-\pi_n\mathcal{K}$ is also invertible for all sufficiently large $n \geq n_0$, and

   $$C_4 = \sup_{n\geq n_0} \left\|(\mathcal{I}-\pi_n\mathcal{K})^{-1}\right\| < \infty, \tag{1.3.6}$$

where $\mathcal{I}$ denotes the identity operator on $\mathcal{X}$. To derive an error estimate for the approximation φ_n, let φ denote the exact solution of (1.3.1). Then

$$(\mathcal{I}-\mathcal{K})\varphi = f.$$

Applying π_n to both sides, we obtain

$$(\mathcal{I}-\pi_n\mathcal{K})\varphi = \pi_n f + (\varphi - \pi_n\varphi).$$

Since the approximate solution φ_n satisfies

$$(\mathcal{I}-\pi_n\mathcal{K})\varphi_n = \pi_n f,$$

it follows that

$$(\mathcal{I}-\pi_n\mathcal{K})(\varphi-\varphi_n) = \varphi - \pi_n\varphi,$$

and hence

$$\varphi-\varphi_n = (\mathcal{I}-\pi_n\mathcal{K})^{-1}(\varphi - \pi_n\varphi).$$

Taking norms and using estimate (1.3.6), we obtain

$$\|\varphi-\varphi_n\|_\infty \le \left\|(\mathcal{I}-\pi_n\mathcal{K})^{-1}\right\| \|\varphi-\pi_n\varphi\|_\infty \le C_4 \|(\mathcal{I}-\pi_n)\varphi\|_\infty. \tag{1.3.7}$$

If $\pi_n = \mathcal{P}_n$, the orthogonal projection, or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points on each subinterval of a uniform partition of $[0,1]$, and if $\varphi \in \mathcal{C}^{2r+1}[0,1]$, then

$$\|(\mathcal{I}-\pi_n)\varphi\|_\infty = O\big(h^{2r+1}\big).$$

If $f \in \mathcal{C}^{2r+1}[0,1]$ and $\kappa \in \mathcal{C}^{2r+1}(\Omega)$, then $\varphi \in \mathcal{C}^{2r+1}[0,1]$, and it follows that

$$\|\varphi-\varphi_n\|_\infty = O\big(h^{2r+1}\big).$$

Here, the Gauss points are chosen as the zeros of the Legendre polynomial of degree $2r+1$ mapped onto each subinterval, which ensures optimal interpolation properties and leads to the above order of convergence.

2. **Iterated Galerkin/Iterated Collocation (Sloan) Methods.**

   Consider

$$\varphi_n^S - \mathcal{K}\pi_n\varphi_n^S = f.$$

Since $\mathcal{K}$ is compact and π_n converges pointwise to the identity operator, the sequence of operators $\{\mathcal{K}\pi_n\}$ converges to $\mathcal{K}$ in a collectively compact fashion. Since $\mathcal{I}-\mathcal{K}$ is invertible, it follows that $\mathcal{I}-\mathcal{K}\pi_n$ is invertible for $n \geq n_0$ and

$$C_5 = \sup_{n\geq n_0} \left\|(\mathcal{I}-\mathcal{K}\pi_n)^{-1}\right\| < \infty. \tag{1.3.8}$$

To obtain an error estimate for the approximation φ_n^S, we begin by observing that

$$\varphi - \varphi_n^S = (\mathcal{I}-\mathcal{K})^{-1}f - (\mathcal{I}-\mathcal{K}\pi_n)^{-1}f.$$

Applying the second resolvent identity yields

$$\varphi - \varphi_n^S = (\mathcal{I}-\mathcal{K}\pi_n)^{-1}(\mathcal{K}-\mathcal{K}\pi_n)(\mathcal{I}-\mathcal{K})^{-1}f.$$

Since $(\mathcal{I}-\mathcal{K})^{-1}f = \varphi$, taking norms and using estimate (1.3.8), we obtain

$$\left\|\varphi - \varphi_n^S\right\|_\infty \leq C_5 \left\|\mathcal{K}(\mathcal{I}-\pi_n)\varphi\right\|_\infty. \tag{1.3.9}$$

Note that

$$\varphi_n^S = \mathcal{K}\varphi_n + f.$$

Hence once φ_n is computed, φ_n^S is obtained by applying $\mathcal{K}$ to φ_n. This method was proposed by Sloan [83] and φ_n^S is known as iterated Galerkin/iterated collocation solution.

If $\pi_n = \mathcal{P}_n$ or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points on each subinterval of the uniform partition, and if $\varphi \in \mathcal{C}^{4r+2}[0,1]$, then

$$\left\|\mathcal{K}(\mathcal{I}-\pi_n)\varphi\right\|_\infty = O\big(h^{4r+2}\big).$$

If $f \in \mathcal{C}^{4r+2}[0,1]$ and $\kappa \in \mathcal{C}^{4r+2}(\Omega)$, then $\varphi \in \mathcal{C}^{4r+2}[0,1]$, and it follows that

$$\left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{4r+2}\big).$$

Observe that the one step iteration improves order of convergence from $2r+2$ to $4r+2$. All these above results are established in Atkinson [13].

If the collocation points are not Gauss points, then there is no improvement in the order of convergence as compared to the collocation solution φ_n^C, and

$$\left\|\varphi - \varphi_n^C\right\|_\infty = \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+1}\big).$$

This is shown by an example in Chapter 2.

3. **Modified Projection Method.**

Define

$$\mathcal{K}_n^M = \pi_n \mathcal{K} \pi_n + \pi_n \mathcal{K}(\mathcal{I} - \pi_n) + (\mathcal{I} - \pi_n)\mathcal{K}\pi_n, \tag{1.3.10}$$

which can also be written as

$$\mathcal{K}_n^M = \pi_n \mathcal{K} + \mathcal{K}\pi_n - \pi_n \mathcal{K} \pi_n.$$

The exact equation (1.3.1) is approximated by

$$\varphi_n^M - \mathcal{K}_n^M \varphi_n^M = f, \tag{1.3.11}$$

that is the modified projection method proposed by Kulkarni [48].

Note that

$$\left\|\mathcal{K} - \mathcal{K}_n^M\right\| = \|(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\mathcal{I} - \mathcal{K}_n^M$ is invertible for $n \geq n_0$ and

$$C_6 = \sup_{n \geq n_0} \left\|(\mathcal{I} - \mathcal{K}_n^M)^{-1}\right\| < \infty. \tag{1.3.12}$$

To derive an error bound for the modified approximation φ_n^M, we begin with

$$\varphi - \varphi_n^M = (\mathcal{I} - \mathcal{K})^{-1} f - (\mathcal{I} - \mathcal{K}_n^M)^{-1} f.$$

Applying the second resolvent identity, we have

$$\varphi - \varphi_n^M = (\mathcal{I} - \mathcal{K}_n^M)^{-1}(\mathcal{K} - \mathcal{K}_n^M)(\mathcal{I} - \mathcal{K})^{-1} f.$$

Using $(\mathcal{I} - \mathcal{K})^{-1} f = \varphi$ together with estimate (1.3.12), it follows that

$$\left\|\varphi - \varphi_n^M\right\|_\infty \leq C_6 \, \|(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)\varphi\|_\infty. \tag{1.3.13}$$

If $\pi_n = \mathcal{P}_n$ or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points, and if $\kappa \in \mathcal{C}^{4r+2}(\Omega)$ and $\varphi \in \mathcal{C}^{4r+2}[0,1]$, then

$$\|(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)\varphi\|_\infty = O\big(h^{6r+3}\big)\,.$$

It follows that for $f \in \mathcal{C}^{4r+2}[0,1]$ and $\kappa \in \mathcal{C}^{4r+2}(\Omega)$,

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{6r+3}\big)\,.$$

If the collocation points are not Gauss points, then

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{4r+2}\big)\,.$$

4. **Iterated Modified Projection Method.**

   Define
$$\widetilde{\varphi}_n^M = \mathcal{K}\varphi_n^M + f.$$
   This represents a single step iteration applied to the modified approximation φ_n^M. Here, $\widetilde{\varphi}_n^M$ is known as the iterated modified Galerkin/collocation solution.

   We now estimate the error in this refined approximation. Using the definition of $\widetilde{\varphi}_n^M$, we write
$$\begin{aligned}\varphi - \widetilde{\varphi}_n^M &= \mathcal{K}(\varphi - \varphi_n^M)\\ &= \mathcal{K}(\mathcal{I}-\mathcal{K})^{-1}(\mathcal{K}-\mathcal{K}_n^M)(\mathcal{I}-\mathcal{K}_n^M)^{-1}f\\ &= (\mathcal{I}-\mathcal{K})^{-1}\mathcal{K}(\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n)\big(\varphi+\varphi_n^M-\varphi\big),\end{aligned}$$
   Taking norms on both sides gives
$$\begin{aligned}\big\|\varphi - \widetilde{\varphi}_n^M\big\|_\infty \leq & \big\|(\mathcal{I}-\mathcal{K})^{-1}\big\|\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty\\ &+\big\|(\mathcal{I}-\mathcal{K})^{-1}\big\|\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\|\big\|\varphi-\varphi_n^M\big\|_\infty. \end{aligned} \tag{1.3.14}$$
   This inequality shows that the refinement error consists of two contributions: the action of the operator $\mathcal{K}(\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n)$ on the exact solution and the amplification of the previously obtained approximation error $\big\|\varphi-\varphi_n^M\big\|_\infty$.

   If $\pi_n = \mathcal{P}_n$ or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points, and if $\kappa \in \mathcal{C}^{4r+2}(\Omega)$ and $\varphi \in \mathcal{C}^{4r+2}[0,1]$, then
$$\|\mathcal{K}(\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n)\varphi\|_\infty = O\big(h^{8r+4}\big)\,, \quad \|\mathcal{K}(\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n)\| = O\big(h^{4r+2}\big)\,.$$
   It follows that for $f \in \mathcal{C}^{4r+2}[0,1]$ and $\kappa \in \mathcal{C}^{4r+2}(\Omega)$,
$$\big\|\varphi - \widetilde{\varphi}_n^M\big\|_\infty = O\big(h^{8r+4}\big)\,.$$
   All the above results of the two proposed methods are established in Kulkarni [48].

We recall the modified finite-rank approximation of the integral operator $\mathcal{K}$ from (1.3.10) and derive the resulting projection-based algebraic system. The modified operator is given as
$$\mathcal{K}_n^M = \pi_n\mathcal{K}\pi_n + \pi_n\mathcal{K}(\mathcal{I}-\pi_n) + (\mathcal{I}-\pi_n)\mathcal{K}\pi_n.$$
The corresponding finite-rank approximation of the integral equation (1.3.11) is given as
$$\varphi_n^M - \big(\pi_n\mathcal{K}\pi_n + \pi_n\mathcal{K}(\mathcal{I}-\pi_n) + (\mathcal{I}-\pi_n)\mathcal{K}\pi_n\big)\varphi_n^M = f. \tag{1.3.15}$$

To reduce equation (1.3.15) to a system of linear algebraic equations, we apply π_n and $(\mathcal{I} - \pi_n)$ to obtain

$$\pi_n \varphi_n^M - \pi_n \mathcal{K} \pi_n \varphi_n^M - \pi_n \mathcal{K}(\mathcal{I} - \pi_n)\varphi_n^M = \pi_n f, \tag{1.3.16}$$

$$(\mathcal{I} - \pi_n)\varphi_n^M - (\mathcal{I} - \pi_n)\mathcal{K}\pi_n \varphi_n^M = (\mathcal{I} - \pi_n) f. \tag{1.3.17}$$

Let $w_n^M = \pi_n \varphi_n^M$. Using equation (1.3.17) to eliminate $(\mathcal{I} - \pi_n)\varphi_n^M$ from (1.3.16), we obtain the reduced equation

$$w_n^M - \big(\pi_n \mathcal{K} \pi_n + \pi_n \mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}\pi_n\big) w_n^M = \pi_n f + \pi_n \mathcal{K}(\mathcal{I} - \pi_n) f. \tag{1.3.18}$$

Equation (1.3.18) constitutes a closed system for the unknown w_n^M.

Once w_n^M is determined, the approximation φ_n^M is reconstructed as

$$\varphi_n^M = w_n^M + (\mathcal{I} - \pi_n)\mathcal{K} w_n^M + (\mathcal{I} - \pi_n) f,$$

and the corresponding iterated approximation is given by

$$\widetilde{\varphi}_n^M = \mathcal{K}\varphi_n^M + f.$$

Since the dimension of the range $\mathcal{R}(\pi_n) = \mathcal{X}_n$ is $n(2r+1)$, then equation (1.3.18) reduces to a linear system of size $n(2r+1)$. Hence, the size of the algebraic system to be solved is the same as that arising in the standard Galerkin/collocation method. The additional computational effort in the present approach stems from the construction of the matrix associated with the operator $\pi_n \mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}\pi_n$ and from the evaluation of the vectors $\pi_n \mathcal{K}(\mathcal{I} - \pi_n) f$ and $\mathcal{K} w_n^M$. This increase in computational cost is relatively modest and is compensated by the improved order of convergence achieved by the proposed method.

Galerkin and collocation methods approximate the solution of a Fredholm integral equation by reducing it to a finite system of linear equations. Improved approximations, referred to as the iterated Galerkin and iterated collocation solutions, are obtained by applying the integral operator $\mathcal{K}$ to the corresponding approximate solutions. In practical implementations, the inner product as well as integrals arising in these formulations are evaluated using numerical quadrature rules, leading to discrete Galerkin and discrete collocation methods and their iterated variants. For an appropriate choice of quadrature, the discrete iterated Galerkin method coincides with the Nyström method. Similar considerations apply to modified projection methods, where iteration and discretization together yield enhanced accuracy.

We now turn our attention to the case where the kernel κ is of Green's function type, which arises naturally in the numerical treatment of boundary value problems and constitutes an important class of kernels in the theory of Fredholm integral equations of the second kind. Recall that $\Omega = [0,1] \times [0,1]$ and let us consider the partition

$$\Omega_1 = \{(s,t) : 0 \leq t \leq s \leq 1\}, \quad \Omega_2 = \{(s,t) : 0 \leq s \leq t \leq 1\}.$$

A kernel $\kappa : \Omega \to \mathbb{R}$ is said to be of Green's function type if it is defined piecewise by

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t), & (s,t) \in \Omega_1, \\ \kappa_2(s,t), & (s,t) \in \Omega_2, \end{cases}$$

where $\kappa_1 \in \mathcal{C}^{\alpha}(\Omega_1)$, $\kappa_2 \in \mathcal{C}^{\alpha}(\Omega_2)$ for some integer $\alpha \geq 0$, and the condition $\kappa_1(s,s) = \kappa_2(s,s)$ holds. The kernel κ is continuous on Ω but may exhibit reduced smoothness along the diagonal $s = t$, which has a direct impact on the convergence behaviour of numerical approximation methods.

Let $\mathcal{X}_n$ be the space of piecewise polynomials of even degree at most $2r$ defined on a uniform partition of $[0,1]$ with mesh size $h = \dfrac{1}{n}$. Let $\pi_n = \mathcal{P}_n$, the the restriction to $\mathcal{L}^{\infty}[0,1]$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathcal{X}_n$. Chatelin-Lebbar [21] established the following orders of convergence for Galerkin approximation φ_n^G and the corresponding iterated Galerkin solution φ_n^S. The iterated solution get the improved convergence rate. If $r = 0$, then

$$\left\|\varphi - \varphi_n^G\right\|_\infty = O(h)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^2\big)\,,$$

whereas if $r \geq 1$, then

$$\left\|\varphi - \varphi_n^G\right\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+3}\big)\,.$$

Further improvement is obtained by the modified projection method introduced by Kulkarni [51]. In this case, the modified Galerkin solution φ_n^M and its iterated version $\widetilde{\varphi}_n^M$ satisfy the following error estimates. If $r = 0$, then

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^3\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^4\big)\,,$$

whereas if $r \geq 1$, then

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{2r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{2r+5}\big)\,.$$

Thus, φ_n^S converges to φ faster than φ_n^G and $\widetilde{\varphi}_n^M$ converges to φ faster than φ_n^S and φ_n^M.

In this case, let $\pi_n = \mathcal{Q}_n$ denote the interpolatory projection at $2r+1$ Gauss points on each subinterval of a uniform partition of $[0,1]$. Then the following orders of convergence for the collocation solution φ_n^C and the corresponding iterated collocation solution φ_n^S were established by Chatelin-Lebbar [21]. If $r=0$, then

$$\|\varphi - \varphi_n^C\|_\infty = O(h)\,, \quad \|\varphi - \varphi_n^S\|_\infty = O\big(h^2\big)\,,$$

whereas if $r \geq 1$, then

$$\|\varphi - \varphi_n^C\|_\infty = O\big(h^{2r+1}\big)\,, \quad \|\varphi - \varphi_n^S\|_\infty = O\big(h^{2r+3}\big)\,.$$

For the modified collocation method, the following convergence orders are obtained from Kulkarni [51]. If $r=0$, then

$$\|\varphi - \varphi_n^M\|_\infty = O\big(h^3\big)\,,$$

whereas if $r \geq 1$, then

$$\|\varphi - \varphi_n^M\|_\infty = O\big(h^{2r+3}\big)\,.$$

These estimates demonstrate that, although the reduced smoothness of Green's function type kernels along the diagonal limits the attainable orders of convergence, modified projection methods consistently yield higher-order accuracy when compared with the standard Galerkin and collocation methods.

In this thesis, we investigate the convergence behaviour of numerical approximations for Fredholm integral equations of the second kind with both smooth kernels and kernels of Green's function type. The approximate solutions are constructed using the collocation method, the iterated collocation method, and their corresponding modified variants.

In Chapter 2, we introduce a family of finite-dimensional subspaces $\mathcal{X}_n$ consisting of piecewise polynomial functions of even degree at most $2r$ ($r \geq 0$), defined on a uniform partition of $[0,1]$ into n subintervals with mesh size $h = \dfrac{1}{n}$. The projection operator $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ is chosen as an interpolatory projection associated with $2r+1$ equidistant interpolation points on each subinterval $[t_{j-1}, t_j]$, including the endpoints, which are not necessarily Gauss points. For Fredholm integral equations with smooth kernels, we establish that the collocation, iterated collocation, modified collocation, and iterated modified collocation methods converge with orders h^{2r+1}, h^{2r+2}, h^{4r+3}, and h^{4r+4}, respectively. In the case of Green's function type kernels and for $r \geq 1$, the corresponding convergence orders reduce to h^{2r+1}, h^{2r+2}, h^{2r+2}, and h^{2r+3}, reflecting the reduced regularity of the kernel along the diagonal. Moreover, for piecewise constant approximations ($r=0$), the collocation, iterated collocation, modified collocation, and iterated modified collocation methods converge with orders 1, 2, 3, and 4, respectively. These theoretical convergence rates are validated through numerical experiments. Overall, this chapter demonstrates that both iteration and modification lead to substantial improvements in the accuracy of collocation-based methods for Fredholm integral equations of the second kind. In particular, iteration increases the order of convergence, while the modified projection framework yields higher-order superconvergence without increasing the size of the algebraic system. The use of equidistant collocation points together with approximation spaces of even-degree polynomials highlights the robustness and efficiency of the proposed projection framework. Moreover, this choice motivates its further application to associated eigenvalue problems and Urysohn integral equations of the second kind, which is pursued in the subsequent chapter.

1.4 Urysohn Integral Equations

Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$, and let $\Omega \times \mathbb{R} = [0,1] \times [0,1] \times \mathbb{R}$. Consider a continuous kernel $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ and define the Urysohn integral operator by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1], \quad x \in \mathcal{X}. \tag{1.4.1}$$

It is well known that $\mathcal{K}$ is a compact operator from $\mathcal{L}^\infty[0,1]$ into $\mathcal{C}[0,1]$. We are interested in the nonlinear integral equation

$$x - \mathcal{K}(x) = f, \quad f \in \mathcal{X}. \tag{1.4.2}$$

A widely occurring special case of Urysohn integral equations is the Hammerstein integral equation

$$x(s) - \int_0^1 \kappa(s,t)\, g(t, x(t))\,dt = f(s), \quad s \in [0,1],$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Such equations arise naturally when boundary value problems for ordinary or partial differential equations are rewritten in integral form; in this context, the kernel $\kappa(s,t)$ typically represents a Green's function. See Krasnoselskii et al. [45] and Zabreiko-Povolotskii [93].

We assume that the equation (1.4.2) admits a unique solution, denoted by φ. If the kernel $\kappa(s,t,u)$ of $\mathcal{K}$ satisfies $\dfrac{\partial \kappa}{\partial u} \in \mathcal{C}(\Omega \times \mathbb{R})$, then $\mathcal{K}$ is Fréchet differentiable on $\mathcal{X}$ and its derivative at $x \in \mathcal{X}$ is given by

$$(\mathcal{K}'(x)y)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))\, y(t)\, dt, \quad s \in [0,1],\ y \in \mathcal{X},$$

where $\dfrac{\partial \kappa}{\partial u}(s,t,x(t))$ denotes the partial derivative of κ with respect to its third variable, evaluated at $x(t)$. To study the local behaviour of solutions, we rewrite (1.4.2) as

$$F(x) = x - \mathcal{K}(x) - f = 0,$$

where $F : \mathcal{X} \to \mathcal{X}$ is Fréchet differentiable, and

$$F'(\varphi) = \mathcal{I} - \mathcal{K}'(\varphi).$$

If 1 is not an eigenvalue of the compact linear operator $\mathcal{K}'(\varphi)$, then $\mathcal{I} - \mathcal{K}'(\varphi)$ is invertible. By the inverse function theorem in Banach spaces, the equation $F(x) = 0$ admits a unique solution in a neighbourhood of φ. Thus, φ is an isolated solution of (1.4.2).

Since equations of the form (1.4.2) cannot, in general, be solved exactly, we are interested in approximate solutions of the above equation and it is necessary to construct accurate numerical approximations. A common approach is to replace (1.4.2) with a sequence of finite-dimensional approximating problems,

$$x_n - \mathcal{K}_n(x_n) = f_n,$$

where $\mathcal{K}_n$ is a suitable finite-rank approximation of $\mathcal{K}$. The approximate solution x_n is then computed by solving the corresponding nonlinear system using an iterative method. The methods for approximation fall into two categories:

1. **Nyström methods**, in which the integral in (1.4.1) is replaced by a convergent numerical quadrature rule, yielding finite-dimensional Nyström operators; which approximates $\mathcal{K}$. The Nyström method for nonlinear integral equations is investigated in Atkinson [5] and in Weiss [91].

2. **Projection methods**, based on a sequence of continuous finite-rank projections converging pointwise to the identity operator, and use it to define operators that approximate $\mathcal{K}$. The classical Galerkin method and the collocation method, associated respectively with the orthogonal and interpolatory projections, fall in this category. Projection methods are studied by a number researchers [3, 10, 42, 43, 44, 67, 68, 89, 90]. In [31, 32], another method called the modified projection method is defined and

investigated by Grammont and Kulkarni. The discrete versions of the Galerkin and the iterated Galerkin methods are considered in Atkinson-Potra [11]. On the other hand, the discrete versions of the collocation and the iterated collocation methods are considered by Atkinson-Flores in [9]. Furthermore, Kulkarni-Rakshit introduced the discrete versions of the modified projection and of the iterated modified projection methods for Urysohn integral equations and developed rigorous analyses of their convergence and superconvergence properties; see [54, 55].

Both approaches reduce the infinite-dimensional equation (1.4.2) to a finite-dimensional nonlinear system to which standard numerical techniques may be applied. In the subsequent subsections, we focus on projection-based methods and their variants, with particular attention to modified projection methods that yield higher orders of convergence while retaining a reasonable computational cost.

1.4.1 Projection Methods

For $r \geq 0$, let $\mathcal{X}_n$ denote the space of piecewise polynomials of even degree at most $2r$ defined over a uniform partition of $[0,1]$ into n subintervals. With mesh-length $h = \dfrac{1}{n}$, the absence of continuity conditions at the nodes implies that $\dim(\mathcal{X}_n) = n(2r+1)$, and clearly $\mathcal{X}_n \subset \mathcal{L}^\infty[0,1]$. We recall the two types of projections from $\mathcal{L}^\infty[0,1]$ to $\mathcal{X}_n$.

- The map $\mathcal{P}_n$ is the restriction to $\mathcal{L}^\infty[0,1]$ of the orthogonal projection from $\mathcal{L}^2[0,1]$ to $\mathcal{X}_n$.
- Choose $2r+1$ distinct points in each of the subintervals $\Delta_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$. Let $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ be the map which interpolates a given function at the $(2r+1)n$ points in $[0,1]$. Using Hahn-Banach theorem, this map is necessarily extended to $\mathcal{L}^\infty[0,1]$ as in Atkinson et al. [8], and then $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ is a projection.

For convenience, we write $\pi_n = \mathcal{P}_n$ or $\mathcal{Q}_n$, so that the Galerkin and collocation methods can be expressed in a unified form. In both cases,

$$\pi_n x \to x, \quad x \in \mathcal{C}[0,1],$$

and, for any $x \in \mathcal{C}^\alpha[0,1]$, the following estimate holds:

$$\|(\mathcal{I} - \pi_n)x\|_\infty \leq C_7 \|x^{(\beta)}\|_\infty h^\beta,$$

where $\beta = \min\{\alpha,\, 2r+1\}$ and C_7 is a constant independent of n.

Now, we will give varinats of projection methods as follows:

1. **Galerkin and Collocation Methods.**

   The Galerkin approximation of equation (1.4.2) is given by

   $$\varphi_n^G - \mathcal{P}_n \mathcal{K}(\varphi_n^G) = \mathcal{P}_n f.$$

   Similarly, in the collocation method, equation (1.4.2) is approximated by

   $$\varphi_n^C - \mathcal{Q}_n \mathcal{K}(\varphi_n^C) = \mathcal{Q}_n f.$$

   These classical projection methods have been extensively studied in the existing literature. Detailed discussions can be found in the works of Krasnoselskii [42], Krasnoselskii et al. [43], and Krasnoselskii-Zabreiko [44].

2. **Iterated Galerkin/Iterated Collocation Methods.**

   Using one step of iteration, the iterated Galerkin solution is defined as

   $$\varphi_n^S = \mathcal{K}(\varphi_n^G) + f.$$

   In a similar manner, the iterated collocation solution is defined as

   $$\varphi_n^S = \mathcal{K}(\varphi_n^C) + f.$$

   It can be verified that

   $$\varphi_n^S - \mathcal{K}\big(\mathcal{P}_n\varphi_n^S\big) = f \quad \text{or} \quad \varphi_n^S - \mathcal{K}\big(\mathcal{Q}_n\varphi_n^S\big) = f.$$

   The iterated projection methods are analyzed in Atkinson-Potra [10].

   If $\pi_n = \mathcal{P}_n$ or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points in each subinterval $\Delta_j = [t_{j-1}, t_j],\ \ j = 1, 2, \ldots, n$. In case of smooth kernel, assuming that $\dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R})$ and $f \in \mathcal{C}^{2r+1}[0,1]$, Atkinson-Potra [10] established the convergence estimates

   $$\big\|\varphi - \varphi_n^G\big\|_\infty = O\big(h^{2r+1}\big)\,, \quad \big\|\varphi - \varphi_n^C\big\|_\infty = O\big(h^{2r+1}\big)\,, \quad \big\|\varphi - \varphi_n^S\big\|_\infty = O\big(h^{4r+2}\big)\,.$$

   Observe that the one step iteration improves order of convergence from $2r+1$ to $4r+2$. If the collocation points are not Gauss points, then there is no improvement in the order of convergence as compared to the collocation solution φ_n^C. In fact, Atkinson-Potra [10] demonstrate that

$$\left\|\varphi - \varphi_n^C\right\|_\infty = \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+1}\big)\,.$$

This is also shown by an example in Chapter 3. Thus, we observe that selecting Gauss points as collocation points improves the convergence orders through iterative refinement.

3. **Modified Projection Method.**

   Let π_n denote either $\mathcal{P}_n$ or $\mathcal{Q}_n$. In [31, 32], the modified projection approximation of (1.4.2) is defined through

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f, \tag{1.4.3}$$

   where the modified projection operator is given by

$$\mathcal{K}_n^M(x) = \pi_n\mathcal{K}(x) + \mathcal{K}(\pi_n x) - \pi_n\mathcal{K}(\pi_n x), \quad x \in \mathcal{X}. \tag{1.4.4}$$

   The method represents a generalization of the modified projection technique proposed in the linear framework in Kulkarni [48].

4. **Iterated Modified Projection Method.**

   As in the case of the iterated Galerkin/iterated collocation method, we perform one step of iteration of the modified projection solution, and the associated iterated modified projection solution is defined by

$$\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f.$$

   If $\pi_n = \mathcal{P}_n$ or $\pi_n = \mathcal{Q}_n$, the interpolatory projection at $2r+1$ Gauss points in each subinterval $\Delta_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$. Under the stronger smoothness requirements $\dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{4r+2}(\Omega \times \mathbb{R})$ and $f \in \mathcal{C}^{4r+2}[0,1]$, Grammont et al. [33] proved the higher-order bounds as

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{6r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{8r+4}\big)\,.$$

   It is well known that, if the collocation points are not Gauss points, then the collocation method yields an error of order $2r+1$, and a single iteration does not enhance this convergence order (see Atkinson-Potra [10]). In contrast, the modified collocattion method achieves an error of order $4r+2$, thereby providing a significantly improved approximation compared to both the collocation and iterated collocation methods (see Grammont et al. [33]).

Since $\mathcal{X}_n$ is a finite-dimensional subspace of $\mathcal{X}$, the construction of the Galerkin approximation φ_n^G and the collocation approximation φ_n^C reduces to solving a nonlinear system in a finite-dimensional space. An analogous situation holds for the modified projection approximation φ_n^M, although the operator defining φ_n^M differs from those associated with the Galerkin and collocation methods, its evaluation likewise leads to a nonlinear system of the same dimension. It is important to emphasize that, at each discretization level, the linear systems that arise in the implementation of all these methods possess identical size. Hence, the computational complexity per discretization level is essentially comparable for the Galerkin, collocation, and modified projection methods. However, despite this similarity in computational cost, the modified projection method exhibits a significantly superior convergence order. Consequently, the sequence $\{\varphi_n^M\}$ converges to the exact solution at a notably faster rate than the sequences $\{\varphi_n^G\}$ and $\{\varphi_n^C\}$ generated by the Galerkin and collocation methods.

If the kernel κ of the Urysohn integral operator $\mathcal{K}$ defined by (1.4.1) possesses sufficient smoothness, then the Galerkin and collocation methods yield errors of order h^{2r+1}, whereas the corresponding iterated Galerkin and iterated collocation methods achieve higher-order convergence of order h^{4r+2} (see Atkinson-Potra [10]). On the other hand, the modified projection and iterated modified projection methods provide further improvement, with errors of order h^{6r+3} and h^{8r+4}, respectively (see Grammont et al. [33]).

We now turn to the case where the kernel κ is of Green's function type. For this purpose, we partition the domain $\Omega \times \mathbb{R} = [0,1] \times [0,1] \times \mathbb{R}$ into two regions:

$$\Omega_1 \times \mathbb{R} = \{(s,t,u) : 0 \le t \le s \le 1,\ u \in \mathbb{R}\}, \quad \Omega_2 \times \mathbb{R} = \{(s,t,u) : 0 \le s \le t \le 1,\ u \in \mathbb{R}\}.$$

Assume that

$$\frac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega_1 \times \mathbb{R}), \quad \frac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega_2 \times \mathbb{R}), \quad \frac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R}).$$

In Atkinson-Potra [10], the analysis is carried out for the case where the orthogonal projection $\mathcal{P}_n$ is employed, and the associated orders of convergence for the Galerkin solution φ_n^G and the iterated Galerkin solution φ_n^S are derived. For linear integral equations, corresponding results were established earlier by Chatelin-Lebbar [21].

If $r = 0$, then

$$\left\|\varphi - \varphi_n^G\right\|_\infty = O(h)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^2\big)\,,$$

whereas if $r \ge 1$, then

$$\left\|\varphi - \varphi_n^G\right\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+3}\big)\,.$$

In Grammont et al. [34], the following orders of convergence for the modified methods are obtained. If $r = 0$, then

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\left(h^3\right), \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^4\right),$$

whereas if $r \geq 1$, then

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\left(h^{2r+3}\right), \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^{2r+5}\right).$$

The preceding error bound generalizes the result of Kulkarni [51] for linear integral operators with Green's function type kernels.

We now consider the interpolatory projection $\mathcal{Q}_n$ defined at $2r+1$ Gauss points in each subinterval $\Delta_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$. Let $\alpha \geq 0$ and assume that the kernel κ of the Urysohn integral operator introduced in (1.4.1) satisfies

$$\frac{\partial \kappa}{\partial u}, \frac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^\alpha(\Omega_1 \times \mathbb{R}), \quad \frac{\partial \kappa}{\partial u}, \frac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^\alpha(\Omega_2 \times \mathbb{R}), \quad \frac{\partial \kappa}{\partial u} \in \mathcal{C}(\Omega \times \mathbb{R}). \tag{1.4.5}$$

If $\alpha \geq 2r+1$, the orders of convergence for the collocation solution and the iterated collocation solution established by Atkinson-Potra [10] are given by

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\left(h^{2r+1}\right), \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\left(h^{\min(\alpha,\, 4r+2,\, 2r+3)}\right).$$

Along with the regularity conditions in (1.4.5), we assume in addition that

$$\frac{\partial^4 \kappa}{\partial u^4} \in \mathcal{C}(\Omega \times \mathbb{R}).$$

Under this assumption, the following result holds in Kulkarni-Nidhin [57] for the modified collocation method and its iterated version:

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\left(h^{2r+3}\right).$$

When $\alpha \geq 4$ and $r = 0$, we obtain

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^4\right).$$

If $r \geq 1$ and $\alpha \geq 2r+3$, the iterated modified collocation solution satisfies the enhanced estimate

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^{4r+3}\right).$$

A comparison of the above error estimates shows that, when $r = 0$, the orders of convergence corresponding to the collocation, iterated collocation, modified collocation, and iterated modified collocation methods are $1, 2, 3$, and 4, respectively. For $r \geq 1$ and $\alpha \geq 2r+3$, both φ_n^S and φ_n^M converge to the exact solution φ with the same rate, whereas the iterated modified collocation approximation $\widetilde{\varphi}_n^M$ achieves a higher rate of convergence. These bounds generalize the result obtained by Kulkarni [51], which concerns the case of a linear integral operator with a Green's function type kernel.

In this thesis, we also investigate the orders of convergence of approximate solutions to Urysohn integral equations with both smooth kernels and kernels of Green's function type, obtained using the collocation method, iterated collocation method, and their modified variants. In Chapter 3, we consider the finite-dimensional space $\mathcal{X}_n$ consisting of piecewise polynomials of even degree $\leq 2r$ $(r \geq 0)$ defined on a uniform partition of $[0, 1]$ into n subintervals of mesh size $h = \dfrac{1}{n}$. The projection operator $\mathcal{Q}_n : \mathcal{C}[0, 1] \to \mathcal{X}_n$ is taken to be the interpolatory projection at $2r + 1$ equidistant nodes (including the endpoints) on each subinterval $[t_{j-1}, t_j]$; these nodes are not necessarily the Gauss points.

For smooth kernels, we establish that the collocation, iterated collocation, modified collocation, and iterated modified collocation methods converge with orders h^{2r+1}, h^{2r+2}, h^{4r+3}, and h^{4r+4}, respectively. For kernels of Green's function type, and for $r \geq 1$, the corresponding orders of convergence are shown to be h^{2r+1}, h^{2r+2}, h^{2r+2}, and h^{2r+3}, respectively. Additionally, when $r = 0$, the orders of convergence associated with the collocation, iterated collocation, modified collocation, and iterated modified collocation methods reduce to $1, 2, 3$, and 4, respectively. The theoretical orders of convergence which we proved in this thesis are validated by numerical results.

In many recent studies, such as [54, 57, 62, 64], enhanced convergence is often achieved by selecting collocation points as the zeros of certain special functions. Such choices are well known to improve the accuracy of numerical integration and are widely employed in classical iterated collocation methods. In contrast, the present work adopts a simpler selection of collocation points, leading to methods that are easier to implement while still maintaining strong theoretical and numerical performance, thereby illustrating the effectiveness of the proposed projection framework. Chapter 3 extends the results of Chapter 2 to Fredholm integral equations with both smooth kernels and kernels of Green's function type, using an interpolatory projection.

1.4.2 Analytical Preliminaries for Urysohn Integral Operators

In this subsection, we present the basic notions concerning multilinear operators and Fréchet differentiability, as well as the Generalised Taylor's theorem, which will play a central role in the analysis carried out in Chapter 3.

Multilinear Operators

For $k \geq 1$ let $\mathcal{X}^k = \mathcal{X} \times \cdots \times \mathcal{X}$ denote the Cartesian product of k copies of $\mathcal{X}$. For $k = 1, 2, \ldots$, write $\mathcal{BL}(\mathcal{X}^{k+1}, \mathcal{Y})$ for the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{BL}(\mathcal{X}^k, \mathcal{Y})$, with the convention $\mathcal{BL}(\mathcal{X}^1, \mathcal{Y}) = \mathcal{BL}(\mathcal{X}, \mathcal{Y})$. Equivalently, a member of $\mathcal{BL}(\mathcal{X}^{k+1}, \mathcal{Y})$ is a bounded operator from $\mathcal{X}^{k+1}$ into $\mathcal{Y}$. An element of $\mathcal{BL}(\mathcal{X}^k, \mathcal{Y})$ is called a k-linear operator from $\mathcal{X}$ into $\mathcal{Y}$.

If $\mathcal{T}_0 \in \mathcal{BL}(\mathcal{X}^k, \mathcal{Y})$ and $x_1, \ldots, x_k \in \mathcal{X}$, then by definition $\mathcal{T}_0 x_1 \in \mathcal{BL}(\mathcal{X}^{k-1}, \mathcal{Y})$. We therefore write $\mathcal{T}_0 x_1 x_2 \cdots x_{k-1} \in \mathcal{BL}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{T}_0 x_1 x_2 \cdots x_{k-1} x_k \in \mathcal{Y}$. Let $x \in \mathcal{X}$, we denote

$$\mathcal{T}_0 x_1 x_2 \cdots x_{k-1} x_k = \mathcal{T}_0(x_1, x_2, \ldots, x_k) \in \mathcal{Y},$$

and

$$\mathcal{T}_0 x^k = \mathcal{T}_0(\underbrace{x, x, \ldots, x}_{k \text{ times}}).$$

If $\mathcal{T}_0$ is a k-linear operator, it may be viewed as an operator on $\mathcal{X} \times \cdots \times \mathcal{X}$ into $\mathcal{Y}$, and its norm is

$$\|\mathcal{T}_0\| = \sup\left\{\|A_j(\mathcal{T}_1, \ldots, x_j)\| : \ \|x_i\| = 1, \ i = 1, \ldots, k\right\}.$$

Elements of $\mathcal{BL}(\mathcal{X}^k, \mathcal{Y})$ are called multilinear operators; for $k = 2$ they are bilinear, and for $k = 3$ they are trilinear.

For a bilinear operator $\mathcal{T}$ from $\mathcal{X}$ into $\mathcal{BL}(\mathcal{X}, \mathcal{Y})$ and $x_1 \in \mathcal{X}$, the map $\mathcal{T} x_1$ is a bounded linear operator from $\mathcal{X}$ into $\mathcal{Y}$. As a mapping $\mathcal{T} \in \mathcal{BL}(\mathcal{X}^2, \mathcal{Y})$ is given by

$$y = \mathcal{T}(x_1, x_2) = (\mathcal{T} x_1) x_2, \quad x_1, x_2 \in \mathcal{X}.$$

We adopt the convention that a multilinear operator acts on the adjacent point first, that is,

$$\mathcal{T} x_1 x_2 = (\mathcal{T} x_1) x_2, \quad x_1, x_2 \in \mathcal{X}.$$

If $\mathcal{T} x_1 x_2 = \mathcal{T} x_2 x_1$ for all $x_1, x_2 \in \mathcal{X}$, then $\mathcal{T}$ is symmetric. For more details see Rall [73].

Fréchet Differentiability

Let $\mathcal{X}$ and $\mathcal{Y}$ be normed linear spaces and $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ an operator. Recall that the operator $\mathcal{T}$ is said to be Fréchet differentiable at $x_0 \in \mathcal{X}$ if there exists a bounded linear operator $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{\|h\| \to 0} \frac{\|\mathcal{T}(x_0 + h) - \mathcal{T}(x_0) - \mathcal{L}h\|_{\mathcal{Y}}}{\|h\|_{\mathcal{X}}} = 0.$$

In this case $\mathcal{L}$ is the Fréchet derivative of $\mathcal{T}$ at x_0, and we write $\mathcal{T}'(x_0) = \mathcal{L}$.

The Fréchet derivative satisfies the usual linearity and chain-rule properties:

- If $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is linear then $\mathcal{L}'(x_0) = \mathcal{L}$ for every $x_0 \in \mathcal{X}$.
- If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{Y}$ are differentiable at $x_0 \in \mathcal{X}$ and α, β are scalars, then
$$(\alpha\mathcal{S} + \beta\mathcal{T})'(x_0) = \alpha\mathcal{S}'(x_0) + \beta\mathcal{T}'(x_0).$$
- If $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ and $S : \mathcal{Y} \to \mathcal{Z}$ are Fréchet differentiable with $\mathcal{T}$ differentiable at x_0 and $\mathcal{S}$ at $\mathcal{T}(x_0)$, then the chain rule (Linz [59]) yields
$$(\mathcal{S} \circ \mathcal{T})'(x_0) = \mathcal{S}'(\mathcal{T}(x_0))\, \mathcal{T}'(x_0).$$

Recall from (1.4.4) that

$$\mathcal{K}_n^M(x) = \pi_n\mathcal{K}(x) + (\mathcal{I} - \pi_n)\mathcal{K}(\pi_n x).$$

Since π_n is a linear operator, using the above properties we obtain

$$\begin{aligned}(\mathcal{K}_n^M)'(x) &= \pi_n\mathcal{K}'(x) + (\mathcal{I} - \pi_n)\mathcal{K}'(\pi_n x)\, \pi_n \\ &= \pi_n\mathcal{K}'(x) + (\mathcal{I} - \pi_n)\mathcal{K}'(\pi_n x)\pi_n. \qquad (1.4.6)\end{aligned}$$

If the second derivative $\mathcal{T}''(x_0)$ exists, it is a symmetric bilinear operator from $\mathcal{X}$ into $\mathcal{Y}$. In general, for $k \geq 1$ denote by $\mathcal{T}^{(k)}(x_0)$ the k-th Fréchet derivative of $\mathcal{T}$ at x_0; then $\mathcal{T}^{(k)}(x_0)$ is a bounded k-linear operator from $\mathcal{X}^k$ into $\mathcal{Y}$. For $x \in \mathcal{X}$ we write

$$\mathcal{T}^{(k)}(x_0)x^k = \mathcal{T}^{(k)}(x_0)(\underbrace{x, \ldots, x}_{k \text{ times}}).$$

Generalized Taylor's Theorem

For $\delta_0 > 0$ the closed ball centred at x_0 of radius δ_0 is

$$\mathcal{B}(x_0, \delta_0) = \{x \in \mathcal{X} : \ \|x - x_0\| \leq \delta_0\}.$$

If $\mathcal{T}$ is $(k-1)$-times differentiable on a neighbourhood $\mathcal{B}(x_0, \delta_0)$ and there exists a k-linear operator $\mathcal{M}$ such that

$$\lim_{\|h\|\to 0} \frac{\left\|\mathcal{T}^{(k-1)}(x_0 + h) - \mathcal{T}^{(k-1)}(x_0) - \mathcal{M}h\right\|}{\|h\|} = 0,$$

then $\mathcal{M} = \mathcal{T}^{(k)}(x_0)$ is the k-th derivative of $\mathcal{T}$ at x_0.

For $x_0, x_1 \in \mathcal{X}$ let the line segment joining x_0 and x_1 be

$$L(x_0, x_1) = \{x \in \mathcal{X} : \ x = \theta x_1 + (1-\theta)x_0, \ \theta \in [0,1]\}.$$

Theorem 2 (Generalized Taylor's Theorem)**.** *[Rall [73, p. 124]] Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $\mathcal{T} : \mathcal{X} \to \mathcal{Y}$ an operator which is m-times Fréchet differentiable in a neighbourhood $\mathcal{B}(x_0, \delta_0)$, $\delta_0 > 0$, of $x_0 \in \mathcal{X}$. Assume that $\mathcal{T}^{(m)}(x)$ is integrable from 0 to 1 along any $x_1 \in \mathcal{B}(x_0, \delta_0)$. Then for $x_1 \in \mathcal{B}(x_0, \delta_0)$,*

$$\mathcal{T}(x_1) = \mathcal{T}(x_0) + \sum_{k=1}^{m-1} \frac{1}{k!} \mathcal{T}^{(j)}(x_0)(x_1 - x_0)^k + R_m(x_0, x_1),$$

where

$$R_m(x_0, x_1) = \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1}\, \mathcal{T}^{(m)}\big(x_0 + \theta(x_1 - x_0)\big)(x_1 - x_0)^m \, d\theta.$$

Fréchet Derivatives of the Urysohn Operator

We now recall the Fréchet derivatives of the Urysohn integral operator $\mathcal{K}$ defined in (1.4.1). If for $j \geq 1$ the partial derivative $\dfrac{\partial^j \kappa}{\partial u^j} \in \mathcal{C}(\Omega \times \mathbb{R})$, then the j-th Fréchet derivative of $\mathcal{K}$ is given by

$$\big(\mathcal{K}^{(j)}(x)(y_1, \ldots, y_j)\big)(s) = \int_0^1 \frac{\partial^j \kappa}{\partial u^j}\big(s, t, x(t)\big)\, y_1(t) \cdots y_j(t)\, dt, \quad s \in [0,1].$$

Moreover, one has the bound

$$\left\|\mathcal{K}^{(j)}(x)\right\| \leq \max_{s,t\in[0,1],\, u\in\mathbb{R}} \left|\frac{\partial^j \kappa}{\partial u^j}(s, t, u)\right|.$$

In particular, let $y_1, y_2 \in \mathcal{L}^\infty[0,1]$. Assume that $\dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R})$. Then the operator $\mathcal{K}$ is twice Fréchet differentiable and the first two Fréchet derivatives of $\mathcal{K}$ at x are given by

$$(\mathcal{K}'(x)y_1)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))\, y_1(t)\, dt, \quad s \in [0,1],$$

$$(\mathcal{K}''(x)(y_1,y_2))(s) = \int_0^1 \frac{\partial^2 \kappa}{\partial u^2}(s,t,x(t))\, y_1(t)\, y_2(t)\, dt, \quad s \in [0,1].$$

Furthermore, one can see that if 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$, then φ is an isolated solution of (1.4.2) by following Krasnoselskii-Zabreiko [44].

Proposition 3 (Krasnoselskii-Zabreiko [44])**.** *Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$. Let $\mathcal{K}$ be the Urysohn integral operator defined by* (1.4.1)*. Let φ be a solution of the equation* (1.4.2)*. Assume that $\dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R})$ and that 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$. Then φ is an isolated solution of* (1.4.2)*.*

Proof. A detailed proof can be found in [70, Proposition 1.2]. □

1.4.3 Existence of Approximate Solutions

In this section, we first recall a result of Atkinson-Potra [10] concerning the existence of the collocation solution φ_n^C in a neighbourhood of the exact solution φ. We then present the classical Neumann series result together with a perturbation result due to Atkinson [13]. After establishing some preliminary results in Lemmas 1.4.1, 1.4.2, and 1.4.3, one can prove in Theorem 7 the existence and uniqueness of the modified collocation solution φ_n^M in a neighbourhood of φ.

Let us recall that $\mathcal{X} = \mathcal{L}^\infty[0,1]$ and the Urysohn integral operator

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\, dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where the kernel κ is a real valued continuous function defined on $\Omega \times \mathbb{R} = [0,1] \times [0,1] \times \mathbb{R}$ and is such that

$$\frac{\partial \kappa}{\partial u}, \; \frac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R}).$$

Then $\mathcal{K}$ is Fréchet differentiable and its Fréchet derivative is given by

$$(\mathcal{K}'(x)y)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))\, y(t)\, dt, \quad s \in [0,1], \quad y \in \mathcal{X}.$$

Assume that for $f \in \mathcal{C}[0,1]$ the equation

$$x - \mathcal{K}(x) = f$$

has a unique solution φ. Since $\mathcal{K}$ maps $\mathcal{L}^\infty[0,1]$ to $\mathcal{C}[0,1]$, it follows that $\varphi \in \mathcal{C}[0,1]$. Let $\mathcal{Q}_n$ denote the interpolatory projection defined in Section 1.4.1. Then

$$\mathcal{Q}_n x \to x \quad \text{as } n \to \infty \quad \text{for } x \in \mathcal{C}[0,1], \tag{1.4.7}$$

and

$$\sup_n \|\mathcal{Q}_n\| < \infty. \tag{1.4.8}$$

In the following, we recall a well-known result from functional analysis.

Proposition 4 (Neumann Expansion). *Let $\mathcal{X}$ be a Banach space and let $\mathcal{T} \in \mathcal{BL}(\mathcal{X})$. If $z \in \mathbb{C}$ satisfies $\|\mathcal{T}\| < |z|$, then the operator $\mathcal{T} - z\mathcal{I}$ is invertible and*

$$(\mathcal{T} - z\mathcal{I})^{-1} = -\sum_{n=0}^{\infty} \frac{\mathcal{T}^n}{z^{n+1}}.$$

Moreover,

$$\|(\mathcal{T} - z\mathcal{I})^{-1}\| \leq \frac{1}{|z| - \|\mathcal{T}\|}.$$

We now quote the following result from Atkinson [13, Theorem 4.1.1].

Proposition 5. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{BL}(\mathcal{X})$. Assume that $\mathcal{T}_1$ is compact and that $(\mathcal{I} - \mathcal{T}_1)^{-1}$ exists as a bounded linear operator on $\mathcal{X}$.*

(a) If

$$\|\mathcal{T}_1 - \mathcal{T}_2\| < \frac{1}{\|(\mathcal{I} - \mathcal{T}_1)^{-1}\|},$$

then $(\mathcal{I} - \mathcal{T}_2)^{-1}$ exists, is bounded, and

$$\|(\mathcal{I} - \mathcal{T}_2)^{-1}\| \leq \frac{\|(\mathcal{I} - \mathcal{T}_1)^{-1}\|}{1 - \|\mathcal{T}_1 - \mathcal{T}_2\| \, \|(\mathcal{I} - \mathcal{T}_1)^{-1}\|}.$$

(b) If

$$\|(\mathcal{T}_1 - \mathcal{T}_2)\mathcal{T}_2\| < \frac{1}{\|(\mathcal{I} - \mathcal{T}_1)^{-1}\|},$$

then $(\mathcal{I} - \mathcal{T}_2)^{-1}$ exists, is bounded, and

$$\|(\mathcal{I} - \mathcal{T}_2)^{-1}\| \leq \frac{\|(\mathcal{I} - \mathcal{T}_1)^{-1}\| \big(1 + \|(\mathcal{I} - \mathcal{T}_1)^{-1}\mathcal{T}_2\|\big)}{1 - \|(\mathcal{I} - \mathcal{T}_1)^{-1}(\mathcal{T}_1 - \mathcal{T}_2)\mathcal{T}_2\|}.$$

Note that the Fréchet derivative $\mathcal{K}'(\varphi)$ is a compact linear operator on $\mathcal{X}$. Since 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$, it follows that

$$(\mathcal{I} - \mathcal{K}'(\varphi))^{-1} \in \mathcal{BL}(\mathcal{X}).$$

The existence and uniqueness of the collocation solution φ_n^C in a neighbourhood of the exact solution φ, for all sufficiently large n, and their convergence to φ were proved in Krasnoselskii [42] and in Krasnoselskii-Zabreiko [44]. We quote the following result from Atkinson-Potra [10]. (See also Krasnoselskii-Zabreiko [44], p. 326.)

Theorem 6. *If* 1 *is not an eigenvalue of* $\mathcal{K}'(\varphi)$*, then there are two nonnegative sequences* $\{\epsilon_n\}, \{\delta_n\}$*, convergent to* 0*, such that*

$$\frac{1}{\|(\mathcal{I} - \mathcal{K}'(\varphi))\|}(1-\epsilon_n)\|\mathcal{Q}_n\varphi - \varphi\|_\infty \leq \left\|\varphi - \varphi_n^C\right\|_\infty \leq \left\|(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}\right\|(1+\delta_n)\|\mathcal{Q}_n\varphi - \varphi\|_\infty.$$

Thus, the rate of convergence of $\{\varphi_n^C\}$ to φ is the same as that of $\{\mathcal{Q}_n\varphi\}$ to φ. Suppose that $(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}$ exists and is bounded. Then, for sufficiently large n, the error in the Sloan iterate satisfies

$$\left\|\varphi - \varphi_n^S\right\|_\infty \leq C_8 \left\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\right\|_\infty,$$

where C_8 is a constant independent of n. The superconvergence of $\{\varphi_n^S\}$ to φ in the case of interpolatory projection at Gauss points is first studied in Atkinson-Potra [10].

We first prove some preliminary results which are crucial for the existence and uniqueness of modified collocation solution φ_n^M.

Lemma 1.4.1. *Let* $\{\mathcal{Q}_n\}$ *be a sequence of interpolatory projections satisfying* (1.4.7) *and* (1.4.8)*. Then*

$$\left\|\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\right\| \to 0 \quad \text{as } n \to \infty.$$

Proof. From (1.4.4), with $\pi_n = \mathcal{Q}_n$, we have

$$\mathcal{K}_n^M(x) = \mathcal{Q}_n\mathcal{K}(x) + (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{Q}_n x).$$

Hence,

$$\begin{aligned}\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi) &= \mathcal{K}(\varphi) - \mathcal{Q}_n\mathcal{K}(\varphi) - (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{Q}_n\varphi) \\ &= -(\mathcal{I} - \mathcal{Q}_n)\Big(\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\Big) \\ &\quad - (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi).\end{aligned}$$

Define
$$\beta_n = \frac{\|\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\|}{\|\mathcal{Q}_n\varphi - \varphi\|}.$$
By the definition of the Fréchet derivative, $\beta_n \to 0$ as $n \to \infty$.

Since $\mathcal{Q}_n x \to x$ for all $x \in \mathcal{C}[0,1]$ and $\mathcal{K}'(\varphi) : \mathcal{L}^\infty[0,1] \to \mathcal{C}[0,1]$ is a compact linear operator, it follows that

$$\eta_n = \|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)\| \to 0 \quad \text{as } n \to \infty. \tag{1.4.9}$$

Since $\varphi \in \mathcal{C}[0,1]$, we obtain

$$\left\|\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\right\| \leq \Big[(1 + \sup_n \|\mathcal{Q}_n\|)\beta_n + \eta_n\Big]\|\mathcal{Q}_n\varphi - \varphi\| \to 0$$

as $n \to \infty$. This completes the proof. □

Lemma 1.4.2. *Assume that* $\dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R})$, *and let* $\mathcal{N}$ *be a bounded neighbourhood of* φ. *Then the Fréchet derivative* $(\mathcal{K}_n^M)'$ *is Lipschitz continuous in* $\mathcal{N}$.

Proof. Since $\dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R})$, the operator $\mathcal{K}$ is twice Fréchet differentiable. Consequently, its first derivative $\mathcal{K}'$ is Lipschitz continuous in any bounded neighbourhood $\mathcal{N}$ of φ. That is, there exists a constant $\gamma > 0$ such that

$$\|\mathcal{K}'(x) - \mathcal{K}'(y)\| \leq \gamma\|x - y\|_\infty, \quad \text{for all } x, y \in \mathcal{N}.$$

Using the representation of $(\mathcal{K}_n^M)'$ given in (1.4.5), we obtain, for $x, y \in \mathcal{N}$,

$$(\mathcal{K}_n^M)'(x) - (\mathcal{K}_n^M)'(y) = \mathcal{Q}_n\big(\mathcal{K}'(x) - \mathcal{K}'(y)\big) + (\mathcal{I} - \mathcal{Q}_n)\big(\mathcal{K}'(\mathcal{Q}_n x) - \mathcal{K}'(\mathcal{Q}_n y)\big)\mathcal{Q}_n.$$

Taking norms and applying the Lipschitz continuity of $\mathcal{K}'$, we obtain

$$\left\|(\mathcal{K}_n^M)'(x) - (\mathcal{K}_n^M)'(y)\right\| \leq \gamma\Big(\sup_n \|\mathcal{Q}_n\| + (1 + \sup_n \|\mathcal{Q}_n\|)\sup_n \|\mathcal{Q}_n\|^2\Big)\|x - y\|_\infty.$$

Using (1.4.8), the right-hand side is bounded by a constant multiple of $\|x - y\|_\infty$, independent of n. This proves that $(\mathcal{K}_n^M)'$ is Lipschitz continuous in $\mathcal{N}$. □

Lemma 1.4.3. *Assume that* 1 *is not an eigenvalue of* $\mathcal{K}'(\varphi)$. *Then there exists a natural number* n_0 *such that for* $n \geq n_0$, $\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)$ *is invertible and*

$$\left\|(\mathcal{I} - (\mathcal{K}_n^M)'(\varphi))^{-1}\right\| \leq C_9,$$

for some constant $C_9 = 2\|(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}\|$ *independent of* n.

Proof. Recall that

$$(\mathcal{K}_n^M)'(\varphi) = \mathcal{Q}_n\mathcal{K}'(\varphi) + (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n.$$

Hence,

$$\begin{aligned}\mathcal{K}'(\varphi) - (\mathcal{K}_n^M)'(\varphi) &= (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi) - (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n \\ &= (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n) + (\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}'(\varphi) - \mathcal{K}'(\mathcal{Q}_n\varphi))\mathcal{Q}_n.\end{aligned}$$

By (1.4.7), (1.4.8), (1.4.9) and the Lipschitz continuity of $\mathcal{K}'$, it follows that

$$\left\|\mathcal{K}'(\varphi) - (\mathcal{K}_n^M)'(\varphi)\right\| \leq (1 + \sup_n \|\mathcal{Q}_n\|)\eta_n + (1 + \sup_n \|\mathcal{Q}_n\|) \sup_n \|\mathcal{Q}_n\|\gamma\|\varphi - \mathcal{Q}_n\varphi\| \to 0,$$

as $n \to \infty$. Choose n_0 such that for $n \geq n_0$,

$$\left\|\mathcal{K}'(\varphi) - (\mathcal{K}_n^M)'(\varphi)\right\| \left\|(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}\right\| \leq \frac{1}{2}.$$

Since $\mathcal{I} - \mathcal{K}'(\varphi)$ is invertible, the Neumann series argument implies that $\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)$ is invertible for all $n \geq n_0$. Moreover, by Proposition 5,

$$\left\|(\mathcal{I} - (\mathcal{K}_n^M)'(\varphi))^{-1}\right\| \leq 2\left\|(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}\right\|.$$

This completes the proof.. □

For $n \geq n_0$, define

$$B_n(x) = x - \left[\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)\right]^{-1} \left[x - \mathcal{K}_n^M(x) - f\right].$$

Then $B_n(x) = x$ if and only if $x - \mathcal{K}_n^M(x) = f$.

For $\delta_0 > 0$, let

$$\mathcal{B}(\varphi, \delta_0) = \{x \in \mathcal{X} : \ \|x - \varphi\|_\infty \leq \delta_0\}.$$

One can prove that, for n large enough, there exists a unique modified collocation solution φ_n^M in a neighbourhood of the exact solution φ. The proof is similar to that of Grammont [31] and is therefore omitted.

Theorem 7. *Assume that the kernel κ of the Urysohn integral operator $\mathcal{K}$, defined by* (1.4.1)*, satisfies* $\dfrac{\partial^2\kappa}{\partial u^2} \in \mathcal{C}(\Omega \times \mathbb{R})$*, and that* 1 *is not an eigenvalue of $\mathcal{K}'(\varphi)$, where φ is the unique solution of the nonlinear integral equation* (1.4.2)*. Then there exists a constant $\delta_0 > 0$ such that, for all sufficiently large n, the modified projection operator*

$\mathcal{K}_n^M$ admits a unique solution $\varphi_n^M \in \mathcal{B}(\varphi, \delta_0)$ of the equation (1.4.3). *Also, there exists a nonnegative sequence $\{\alpha_n\}$, converging to zero as $n \to \infty$, such that the error estimate*

$$\frac{2}{3}\,\alpha_n \leq \left\|\varphi - \varphi_n^M\right\|_\infty \leq 2\,\alpha_n$$

holds, where $\alpha_n = \left\|\left[\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)\right]^{-1}\left[\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\right]\right\|$.

Remark 1. *The solution φ_n^M can equivalently be characterized as the unique fixed point of a contraction mapping B_n on $\mathcal{B}(\varphi, \delta_0)$, where $B_n(x) = x$ if and only if $x - \mathcal{K}_n^M(x) = f$.*

The above result is needed in Chapter 3 to obtain the order of convergence of the modified collocation solution.

1.5 Eigenvalue Problems

This thesis also addresses the numerical approximation of eigenvalue problems associated with Fredholm integral equations of the second kind. In particular, we consider compact integral operators $\mathcal{K}$ defined on the spaces $\mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$, generated by continuous kernel functions. The primary objective is to approximate nonzero eigenvalues and their corresponding eigenfunctions. Owing to the compactness of $\mathcal{K}$, the spectrum consists of isolated eigenvalues with finite multiplicity, which makes projection-based numerical methods especially suitable for such problems.The classical, modified, and their iterated numerical methods studied in this thesis are constructed using orthogonal and interpolatory projections. These approaches reduce the infinite-dimensional eigenvalue problem to finite-dimensional matrix eigenvalue problems while preserving the essential spectral characteristics of the operator. In particular, modified projection methods yield higher-order accuracy by constructing appropriate finite-rank approximations of $\mathcal{K}$, with increasing the dimension of the resulting algebraic systems as compared to standard projection methods.

The convergence analysis is carried out using tools from operator theory, notably spectral projections associated with isolated eigenvalues. This framework allows a clear understanding of how approximate eigenvalues and eigenspaces converge to the exact ones and forms the basis for the error estimates and numerical results presented in the subsequent chapters.

Let $\mathcal{X}$ be a Banach space equipped with the norm $\|\cdot\|$ (in particular, $\mathcal{X} = \mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$), and let $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ be a compact linear operator. The associated eigenvalue problem is to determine eigenvalue $\lambda \in \mathbb{C}$ and a nonzero eigenfunction $\psi \in \mathcal{X}$ such that

$$\mathcal{K}\psi = \lambda\psi, \quad \lambda \neq 0, \quad \|\psi\| = 1.$$

Owing to the compactness of $\mathcal{K}$, its spectrum consists of isolated eigenvalues of finite multiplicity, which makes such problems suitable for numerical approximation.

In this thesis, we focus on compact integral operators of the form

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where κ is a real-valued continuous kernel on $\Omega = [0,1] \times [0,1]$, including kernels of smooth or Green's function type. A standard approach is to approximate $\mathcal{K}$ by a sequence of finite-rank operators $\{\mathcal{K}_n\}$, thereby reducing the problem to finite-dimensional eigenvalue problems which can be treated as matrix eigenvalue problems. The convergence analysis of such approximations is well established; see Ahues et al. [1], Atkinson [13], and Osborn [65]. Several numerical techniques have been developed for constructing the approximating operators $\mathcal{K}_n$. These include Nyström methods, which replace the integral operator by a convergent quadrature rule, and projection-based methods such as Galerkin and collocation methods. A general abstract framework for these methods was given by Atkinson [13]. The convergence of Nyström methods was rigorously analyzed using the concept of collectively compact convergence introduced by Anselone [4] and further studied by Atkinson [6]. Degenerate kernel methods were investigated by Schafer [78], and Nelakanti [63]. Projection methods have a long history in eigenvalue analysis, originating from the work of Kantorovitch [40] and Kantorovitch and Akilov [41]. Their convergence properties are well documented in Krasnoselskii et al. [43] and in the monographs by Baker [15], Chatelin [20], and Ahues et al. [1].

Beyond convergence, the rate of convergence of approximate eigenvalues and eigenfunctions is of primary importance. For smooth kernels, convergence rates and asymptotic expansions have been studied in Ahues et al. [1], Atkinson [13], Baker [15], Chatelin [20], Sloan [86], and Kulkarni [47]. Eigenvalue problems involving Green's kernels, which may lack smoothness along the diagonal, were investigated in Chatelin-Lebbar [22], Rane [74, 76], and Kulkarni-Rane [56]. Superconvergence phenomena for iterated eigenfunctions were first observed by Sloan [84] and further developed by deBoor-Swartz [26, 27] and Chatelin [18, 19, 20].

Further improvements in spectral approximations have been achieved through iterative and modified projection techniques. In particular, the modified projection method proposed by Kulkarni [49, 52] yields faster convergence of eigenelements for sufficiently smooth kernels when compared with classical iterated projection methods, with additional enhancement obtained via the iterated modified projection method. High-order and superconvergent Nyström and degenerate kernel methods based on projections at Gauss points were developed in [2]. In all these approaches, the continuous eigenvalue problem is ultimately reduced to a finite-dimensional matrix eigenvalue problem constructed using suitable piecewise polynomial approximation spaces.

1.5.1 Projection Methods for Eigenvalue Problems

Let $\mathcal{X}$ be a Banach space and recall that $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ be a compact integral operator. We consider the eigenvalue problem

$$\mathcal{K}\psi = \lambda\psi, \tag{1.5.1}$$

where $\lambda \in \mathbb{C} \setminus \{0\}$ and $0 \neq \psi \in \mathcal{X}$. Since $\mathcal{K}$ is compact, its spectrum consists of at most countably many eigenvalues of finite algebraic multiplicity, with zero as the only possible accumulation point. A standard approach for approximating (1.5.1) is to replace $\mathcal{K}$ by a sequence of finite-rank operators and to solve the resulting matrix eigenvalue problems. Let $\{\mathcal{K}_n\}$ be a sequence of finite-rank operators converging to $\mathcal{K}$ in the sense of collective compactness, that is, $\{\mathcal{K}_n x : \|x\| \leq 1, n \in \mathbb{N}\}$ has compact closure in $\mathcal{X}$. Then each $\mathcal{K}_n$ has a finite spectrum, and the eigenvalue problem

$$\mathcal{K}_n\psi_n = \lambda_n\psi_n, \quad \lambda_n \neq 0, \quad \|\psi_n\| = 1$$

reduces to a matrix eigenvalue problem. Under standard assumptions, the approximate eigenpairs (λ_n, ψ_n) converge to a true eigenpair (λ, ψ) of (1.5.1); see Atkinson [13] and Ahues et al. [1]. Projection methods form one of the most widely used classes of such approximations.

For a fixed integer $r \geq 0$, let $\mathcal{X}_n$ denote the space of piecewise polynomials of even degree at most $2r$ defined on a uniform partition of $[0, 1]$ into n subintervals. No continuity conditions are imposed at the partition points. If $h = \dfrac{1}{n}$ denotes the mesh size, then each subinterval contributes $2r+1$ degrees of freedom, and hence $\dim(\mathcal{X}_n) = n(2r+1)$. Clearly, $\mathcal{X}_n \subset \mathcal{L}^\infty[0, 1]$.

We consider two different projections from $\mathcal{L}^\infty[0, 1]$ onto $\mathcal{X}_n$. The first one, denoted by $\mathcal{P}_n$, is obtained by restricting the orthogonal projection from $\mathcal{L}^2[0, 1]$ onto $\mathcal{X}_n$ to the space $\mathcal{L}^\infty[0, 1]$. The second projection, denoted by $\mathcal{Q}_n$, is defined through interpolation. More precisely, $2r+1$ distinct interpolation points are chosen in each subinterval $\Delta_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$, and $\mathcal{Q}_n : \mathcal{C}[0, 1] \to \mathcal{X}_n$ interpolates a given function at these $(2r + 1)n$ points. By an application of the Hahn-Banach theorem (see Atkinson et al. [8]), this operator can be extended to a bounded projection from $\mathcal{L}^\infty[0, 1]$ onto $\mathcal{X}_n$.

For notational convenience, we write $\pi_n = \mathcal{P}_n$ or $\mathcal{Q}_n$, so that both the Galerkin and the collocation methods can be treated in a unified manner. In either case, $\{\pi_n\}_{n\geq 1}$ be a sequence of finite-rank projection operators on $\mathcal{X}$ satisfies

$$\pi_n x \to x, \quad \text{for all } x \in \mathcal{C}[0,1], \quad \text{as } n \to \infty.$$

The classical Galerkin/collocation approximation replaces $\mathcal{K}$ by $\pi_n\mathcal{K}$, leading to the finite-dimensional eigenvalue problem

$$\pi_n \mathcal{K}\psi_n = \lambda_n \psi_n, \quad \psi_n \in \mathcal{R}(\pi_n).$$

A refinement of this approach was proposed by Sloan [84] in the form of the iterated Galerkin/collocation method, where the operator is replaced by $\mathcal{K}\pi_n$. The refined eigenvector is defined as

$$\psi_n^S = \frac{1}{\lambda_n}\mathcal{K}\psi_n.$$

Since φ_n^S is obtained by applying $\mathcal{K}$, it belongs to the range of the original operator, where the exact eigenfunctions lie. Moreover, it is easy to verify that

$$\mathcal{K}\big(\pi_n\psi_n^S\big) = \lambda_n\psi_n^S.$$

This iteration often improves the accuracy of the computed eigenfunctions without increasing the dimension of the discrete eigenvalue problem. These methods are extensively studied in the literature (see, for example, [13, 15, 20, 22, 26, 27, 86]).

To further improve convergence, Kulkarni [49] introduces a modified finite-rank operator

$$\mathcal{K}_n^M = \pi_n\mathcal{K}\pi_n + \pi_n\mathcal{K}(\mathcal{I}-\pi_n) + (\mathcal{I}-\pi_n)\mathcal{K}\pi_n.$$

A straightforward computation shows that

$$\mathcal{K} - \mathcal{K}_n^M = (\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n),$$

which converges to zero in operator norm as $n \to \infty$ under standard approximation assumptions on π_n. Hence, $\{\mathcal{K}_n^M\}$ provides a norm-convergent sequence of finite-rank operators. The corresponding eigenvalue problem is

$$\mathcal{K}_n^M\psi_n^M = \lambda_n^M\psi_n^M, \quad \big\|\psi_n^M\big\| = 1. \tag{1.5.2}$$

Since $\mathcal{K}_n^M$ is finite-rank, (1.5.2) reduces to a matrix eigenvalue problem.

The corresponding iterated modified eigenvector is given as

$$\widetilde{\psi}_n^M = \frac{1}{\lambda_n^M}\,\mathcal{K}\psi_n^M, \tag{1.5.3}$$

which lies in the exact range of $\mathcal{K}$ and provides an improved approximation to the true eigenfunction.

When λ is a simple eigenvalue of $\mathcal{K}$, the modified projection method exhibits a substantially higher order of convergence than the classical projection and iterated projection methods. Since $\mathcal{K}_n^M = \pi_n\mathcal{K} + \mathcal{K}\pi_n - \pi_n\mathcal{K}\pi_n$, and π_n maps $\mathcal{X}$ onto the finite-dimensional space $\mathcal{X}_n$, it follows that for any $x \in \mathcal{X}$, the terms $\pi_n\mathcal{K}x$ and $\pi_n\mathcal{K}\pi_n x$ belong to $\mathcal{X}_n$, while $\mathcal{K}\pi_n x \in \mathcal{K}(\mathcal{X}_n)$. Therefore, $\mathcal{R}(\mathcal{K}_n^M) \subset Y_n := \mathcal{X}_n + \mathcal{K}(\mathcal{X}_n)$. Since $\dim(\mathcal{X}_n) = n(2r+1)$ and $\dim\big(\mathcal{K}(\mathcal{X}_n)\big) \le \dim(\mathcal{X}_n)$, we obtain $\dim(Y_n) \le 2n(2r+1)$. Therefore, the eigenvalue analysis of $\mathcal{K}_n^M$ can be restricted to the finite-dimensional space Y_n, and the associated eigenvalue problem reduces to a matrix eigenvalue problem of dimension at most $2n(2r+1)$. In contrast, the standard projection (or Sloan) method based on $\pi_n\mathcal{K}$ gives a matrix eigenvalue problem of dimension $n(2r+1)$. Compared with this standard projection methods, the modified projection method leads to a quadratic eigenvalue problem, which, after linearization, results in a matrix problem of approximately twice the size of the standard projection methods, thereby increasing the computational cost moderately. However, this additional cost is compensated by the higher-order convergence achieved by the modified projection method, leading to improved accuracy for the same mesh size.

In practical computations, the continuous eigenvalue problems are replaced by fully discrete methods obtained by approximating the integrals appearing in the definition of the operator and in the associated inner products using suitable numerical quadrature formulas. This discretization leads to matrix eigenvalue problems that can be efficiently implemented on a computer. When the projection π_n is chosen as an orthogonal projection, the resulting methods correspond to the discrete Galerkin and iterated discrete Galerkin methods. On the other hand, if π_n is taken to be an interpolatory projection, one obtains the discrete collocation and iterated discrete collocation methods.

The convergence and accuracy of these discrete projection methods for one-dimensional eigenvalue problems with smooth kernels were studied by Kulkarni-Nelakanti [53]. It was shown that, provided the quadrature rules are sufficiently accurate, the discrete methods preserve the convergence rates of the corresponding continuous methods and may even exhibit superconvergence of the approximate eigenvalues and eigenfunctions. Further enhancements in eigenvalue accuracy, based on Richardson extrapolation applied to iterated discrete Galerkin and collocation methods, were reported by Chen et al. [23]. Although, our thesis is focused on projection-based methods only not the discrete projection based methods.

However, the present work is confined to projection-based methods formulated at the operator level and does not involve discretization through numerical quadrature.

1.5.2 Spectral Theory of Compact Operators

Let $\mathcal{X}$ be a Banach space, which in the present thesis is taken to be either $\mathcal{X} = \mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$. Recall that, we denote by $\mathcal{BL}(\mathcal{X})$ the space of all bounded linear operators mapping $\mathcal{X}$ into itself. This space is equipped with the operator norm defined by

$$\|\mathcal{K}\| = \sup_{\|x\|=1} \|\mathcal{K}x\|, \quad \mathcal{K} \in \mathcal{BL}(\mathcal{X}).$$

Let $\mathcal{K} \in \mathcal{BL}(\mathcal{X})$ be a compact linear operator. By definition, $\mathcal{K}$ maps bounded subsets of $\mathcal{X}$ into relatively compact sets. Compact operators arise naturally in the theory of integral equations and play a central role in spectral approximation theory due to their well-structured spectral properties.

Recall that the resolvent set of $\mathcal{K}$ is defined by

$$\rho(\mathcal{K}) = \big\{z \in \mathbb{C} : (\mathcal{K} - z\mathcal{I})^{-1} \in \mathcal{BL}(\mathcal{X})\big\},$$

where $\mathcal{I}$ denotes the identity operator on $\mathcal{X}$. The spectrum of $\mathcal{K}$ is defined as the complement of the resolvent set, namely, $\sigma(\mathcal{K}) = \mathbb{C} \setminus \rho(\mathcal{K})$.

The spectral radius of the operator $\mathcal{K}$ is given by $r_\sigma(\mathcal{K}) = \sup\{|\lambda| : \lambda \in \sigma(\mathcal{K})\}$. Since $r_\sigma(\mathcal{K}) \leq \|\mathcal{K}\|$ and the resolvent set $\rho(\mathcal{K})$ is open, it follows that the spectrum $\sigma(\mathcal{K})$ is a compact subset of $\mathbb{C}$. When $\mathcal{K}$ is a compact operator on an infinite-dimensional Banach space, the spectrum possesses a particularly simple structure. In this case, every nonzero spectral value of $\mathcal{K}$ is an eigenvalue of finite multiplicity, and zero is the only possible accumulation point of the spectrum $\sigma(\mathcal{K})$. The point spectrum of $\mathcal{K}$ consists of all $\lambda \in \sigma(\mathcal{K})$ for which the operator $\mathcal{K} - \lambda\mathcal{I}$ is not injective. For such a value λ, there exists a nontrivial vector $x \in \mathcal{X}$ satisfying $\mathcal{K}x = \lambda x$, and λ is called an eigenvalue of $\mathcal{K}$.

For $\mathcal{K} \in \mathcal{BL}(\mathcal{X})$, we denote the range space and the null space of $\mathcal{K}$, respectively, by

$$\mathcal{R}(\mathcal{K}) = \{\mathcal{K}x : x \in \mathcal{X}\} \quad \text{and} \quad \mathcal{N}(\mathcal{K}) = \{x \in \mathcal{X} : \mathcal{K}x = 0\}.$$

Associated with each eigenvalue λ of the operator $\mathcal{K}$ are two distinct notions of multiplicity. The geometric multiplicity of λ, denoted by $\mathrm{gm}(\lambda)$, is defined as the dimension of the null space of $\mathcal{K}-\lambda\mathcal{I}$, that is, $\mathrm{gm}(\lambda) = \dim \mathcal{N}(\mathcal{K}-\lambda\mathcal{I})$. The algebraic multiplicity of λ is defined as the dimension of the generalized eigenspace associated with λ, which is characterized through the corresponding spectral projection.

The following lemma, due to Anselone [4], provides a useful criterion for upgrading pointwise convergence of bounded operators to uniform convergence on relatively compact subsets of a Banach space.

Lemma 1.5.1. *Let $\mathcal{X}$ be a Banach space and let $\mathcal{Y} \subset \mathcal{X}$ be a relatively compact set. Suppose that $\mathcal{K}$ and $\mathcal{K}_n$ are bounded linear operators on $\mathcal{X}$ satisfying $\|\mathcal{K}_n\| \leq C, \quad$ for all $n \in \mathbb{N}$, where C is a constant independent of n. Assume further that for each $x \in \mathcal{X}$, $\|\mathcal{K}_n x - \mathcal{K}x\| \to 0 \quad$ as $n \to \infty$. Then, $\|\mathcal{K}_n x - \mathcal{K}x\| \to 0 \quad$ as $n \to \infty$, uniformly for all $x \in \mathcal{Y}$. That is, pointwise convergence of $\mathcal{K}_n$ to $\mathcal{K}$ implies uniform convergence on the relatively compact subset $\mathcal{Y}$.*

Let $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ be a sequence of bounded linear operators converging to $\mathcal{K}$ in the collectively compact sense, that is, $\mathcal{K}_n x \to \mathcal{K}x$ for all $x \in \mathcal{X}$ and the set $\{\mathcal{K}_n x : \|x\| \leq 1,\ n \in \mathbb{N}\}$ is relatively compact in $\mathcal{X}$. Such convergence arises naturally in projection-based numerical methods, including Galerkin, collocation, and Nyström methods for integral equations.

Theorem 8. *Let $\mathcal{X}$ be a Banach space and let $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ be a sequence of bounded linear operators converging to $\mathcal{K}$ in the collectively compact sense. Then the following properties hold:*

(i) There exists a constant $C > 0$ such that $\|\mathcal{K}_n\| \leq C$ for all $n \in \mathbb{N}$.

(ii) $\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}\| \to 0$ as $n \to \infty$.

(iii) $\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n\| \to 0$ as $n \to \infty$.

Proof. By the definition of collectively compact convergence, the sequence $\{\mathcal{K}_n\}$ converges pointwise to $\mathcal{K}$ and the set $\mathcal{S} = \{\mathcal{K}_n x : \|x\| \leq 1, n \in \mathbb{N}\}$ is relatively compact in $\mathcal{X}$.

(i) Since $\mathcal{S}$ is relatively compact in a normed space, it is necessarily bounded. Thus, there exists a constant $C > 0$ such that $\|y\| \leq C$ for all $y \in \mathcal{S}$. By the definition of the operator norm,

$$\|\mathcal{K}_n\| = \sup_{\|x\| \leq 1} \|\mathcal{K}_n x\| \leq C$$

for all $n \in \mathbb{N}$, which establishes uniform boundedness.

(ii) Note that $\mathcal{K}$ is a compact operator as it is the pointwise limit of a collectively compact sequence. Let $A = \{\mathcal{K}x : \|x\| \le 1\}$. Since $\mathcal{K}$ is compact, A is relatively compact. A sequence of bounded operators converging pointwise to zero converges uniformly on any relatively compact set, see Lemma 1.5.1. Since $(\mathcal{K}_n - \mathcal{K}) \to 0$ pointwise, it follows that

$$\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}\| = \sup_{\|x\|\le 1} \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}x\| = \sup_{y\in A} \|(\mathcal{K}_n - \mathcal{K})y\| \to 0 \quad \text{as } n \to \infty.$$

(iii) Consider again the set $\mathcal{S}$ as defined at the start of the proof. By hypothesis, $\mathcal{S}$ is relatively compact. Using the same principle used in part (ii), the pointwise convergence of $(\mathcal{K}_n - \mathcal{K})$ to the zero operator implies uniform convergence on the relatively compact set $\mathcal{S}$. Since $\mathcal{K}_n x \in \mathcal{S}$ for all $n \in \mathbb{N}$, it follows that:

$$\|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n\| = \sup_{\|x\|\le 1} \|(\mathcal{K}_n - \mathcal{K})\mathcal{K}_n x\| \le \sup_{y\in \mathcal{S}} \|(\mathcal{K}_n - \mathcal{K})y\|.$$

The right-hand side tends to zero as $n \to \infty$, which completes the proof. $\square$

Here and throughout this section, C denotes a generic positive constant, independent of n, whose value may change from line to line.

Spectral Projections

Let $\lambda \in \sigma(\mathcal{K})$ be an isolated spectral value. Choose a simple, closed, positively oriented contour $\Gamma \subset \rho(\mathcal{K})$ such that λ lies strictly inside Γ and no other point of the spectrum $\sigma(\mathcal{K})$ lies in the interior of Γ. In particular, when $\lambda \neq 0$, the contour Γ can be chosen so that it separates λ from zero and from the remainder of the spectrum of $\mathcal{K}$.

The spectral projection associated with $\mathcal{K}$ and λ is defined by

$$E = -\frac{1}{2\pi i} \int_\Gamma (\mathcal{K} - z\mathcal{I})^{-1}\, dz.$$

The operator E is a bounded projection satisfying $E^2 = E$. Moreover, the range $\mathcal{R}(E)$ coincides with the generalized eigenspace associated with the eigenvalue λ. If $\operatorname{rank} E = m < \infty$, then λ is an eigenvalue of $\mathcal{K}$ and the integer m is called the algebraic multiplicity of λ. In particular, if $\dim \mathcal{R}(E) = 1$, then λ is said to be a simple eigenvalue.

We begin by recalling some standard properties of the spectral projection associated with a simple eigenvalue of a compact operator, which will be used repeatedly in the subsequent analysis.

Proposition 9. *Let $\mathcal{X}$ be a complex Banach space and let $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ be a compact linear operator. Suppose that $\lambda \neq 0$ is a simple eigenvalue of $\mathcal{K}$, with normalized eigenvector ψ satisfying $\mathcal{K}\psi = \lambda\psi$ and $\|\psi\| = 1$. Let Γ be a positively oriented circle enclosing λ and no other point of $\sigma(\mathcal{K})$, and define the spectral projection*

$$E \; = \; -\frac{1}{2\pi i}\int_{\Gamma}(\mathcal{K} - z\mathcal{I})^{-1}\, dz.$$

Then the following statements hold:

(i) $E^2 = E$.

(ii) $\mathcal{R}(E) = \mathcal{N}(\mathcal{K} - \lambda\mathcal{I}) = \operatorname{span}\{\psi\}$.

(iii) $\mathcal{N}(E) = \mathcal{R}(\mathcal{K} - \lambda\mathcal{I})$.

(iv) $\mathcal{X} = \mathcal{R}(E) \oplus \mathcal{N}(E) = \mathcal{N}(\mathcal{K} - \lambda\mathcal{I}) \oplus \mathcal{R}(\mathcal{K} - \lambda\mathcal{I})$, *and every $x \in \mathcal{X}$ admits the unique decomposition $x = Ex + (x - Ex)$.*

(v) $E\mathcal{K} = \mathcal{K}E$.

Remark 2. *Proposition 9 is stated for a simple eigenvalue. If λ has algebraic multiplicity $m > 1$, the spectral projection E still satisfies $E^2 = E$ and $E\mathcal{K} = \mathcal{K}E$, but $\mathcal{R}(E)$ coincides with the generalized eigenspace $\mathcal{N}((\mathcal{K}-\lambda\mathcal{I})^{\ell})$, rather than with the eigenspace itself. Here, ℓ denotes the ascent of the eigenvalue λ, that is, the smallest integer such that $\mathcal{N}((\mathcal{K} - \lambda\mathcal{I})^{\ell}) = \mathcal{N}((\mathcal{K} - \lambda\mathcal{I})^{\ell+1})$.*

For two nonzero subspaces $\mathcal{Y}, \mathcal{Z} \subset \mathcal{X}$, the gap from $\mathcal{Y}$ to $\mathcal{Z}$ is defined by

$$\delta(\mathcal{Y}, \mathcal{Z}) = \sup\bigl\{d(y, \mathcal{Z}) : y \in \mathcal{Y},\ \|y\| = 1\bigr\}, \quad d(y, \mathcal{Z}) = \inf_{z\in\mathcal{Z}} \|y - z\|.$$

Since $\delta(\mathcal{Y}, \mathcal{Z})$ is not symmetric, we define the symmetric gap by

$$\hat{\delta}(\mathcal{Y}, \mathcal{Z}) = \max\{\delta(\mathcal{Y}, \mathcal{Z}), \delta(\mathcal{Z}, \mathcal{Y})\}.$$

This notion provides a quantitative measure of the distance between subspaces and will be used to study convergence of approximate eigenspaces.

Recall that a sequence $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ is converging to $\mathcal{K}$ in the collectively compact sense. By results due to Osborn [65], for all sufficiently large n, the contour Γ lies in $\rho(\mathcal{K}_n)$ and the resolvents $(\mathcal{K}_n - z\mathcal{I})^{-1}$ are uniformly bounded on Γ:

$$\max_{z\in\Gamma} \bigl\|(\mathcal{K}_n - z\mathcal{I})^{-1}\bigr\| < \infty.$$

The spectral projection associated with $\mathcal{K}_n$ is given by

$$E_n = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{K}_n - z\mathcal{I})^{-1}\, dz.$$

Let E_n has rank $m < \infty$, that is, λ is an eigenvalue of $\mathcal{K}$ with algebraic multiplicity m.

Proposition 10 (Ahues et al. [1])**.** *For all sufficiently large n, there exists a constant $C > 0$, independent of n, such that the following estimates hold:*

$$\|E - E_n\| \le C\, \|\mathcal{K} - \mathcal{K}_n\|,$$

$$\|(E - E_n)E\| \le C\, \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}\|,$$

and

$$\|(E - E_n)E_n\| \le C\, \|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}_n\|.$$

We also recall the following result from Ahues et al. [1], p. 86, which plays a crucial role in the analysis of spectral projections.

Proposition 11. *Let P_1 and P_2 be two projections on $\mathcal{X}$. If $r_\sigma(P_1 - P_2) < 1$, then*

$$\operatorname{rank} P_1 = \operatorname{rank} P_2.$$

Using the above estimates, we observe that

$$r_\sigma(E - E_n) \le C\, \|E - E_n\| \to 0 \quad \text{as } n \to \infty,$$

and, moreover,

$$r_\sigma\big((E - E_n)^2\big) \le \big\|(E - E_n)^2\big\| \le \|(E - E_n)E\| + \|(E - E_n)E_n\| \to 0 \quad \text{as } n \to \infty.$$

Consequently, by Proposition 11, it follows that

$$\operatorname{rank} E_n = \operatorname{rank} E = m, \quad \text{for all sufficiently large } n.$$

Thus, for sufficiently large n, E_n has the same rank m as E, and $\mathcal{K}_n$ possesses exactly m eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m}$ inside Γ, counted with algebraic multiplicity. Since individual eigenvalues may reorder under perturbations, it is convenient to consider their arithmetic mean

$$\hat{\lambda}_n = \frac{1}{m}\sum_{j=1}^{m} \lambda_{n,j},$$

which converges to λ as $n \to \infty$. This averaged eigenvalue plays a central role in theoretical error estimates for eigenvalue approximation.

The above framework forms the operator-theoretic foundation for the spectral analysis of compact operators and their numerical approximations. In Chapter 4, it will be applied to integral operators and projection-based methods.

We begin by recalling a fundamental result of Osborn [65], which provides an estimate for the gap between the exact and approximate spectral subspaces associated with a compact operator, and hence measures the closeness of the corresponding eigenfunctions.

Lemma 1.5.2. *For sufficiently large n,*

$$\hat{\delta}(\mathcal{R}(E), \mathcal{R}(E_n)) \leq C\big\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}\big\|,$$

where $(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}$ represents the restriction of $(\mathcal{K} - \mathcal{K}_n)\mathcal{K}$ to $\mathcal{R}(E)$.

Also, we consider a modified result of Osborn [65, Theorem 2], as presented by Kulkarni [52, Theorem 2.2], which provides improved error bounds for the approximation of eigenvalues. Specifically, this result yields the following estimate for the eigenvalue error:

Lemma 1.5.3. *For sufficiently large n,*

$$|\lambda - \hat{\lambda}_n| \leq C\big\|\mathcal{K}_n(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}\big\|.$$

The following result follows from above Lemmas 1.5.2 and 1.5.3 and provides the essential error bounds for the eigenelements.

Theorem 12. *Let $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ be a sequence of bounded linear operators converging to the integral operator $\mathcal{K}$ in the collectively compact sense, with $\mathcal{K}_n = \pi_n\mathcal{K}$. Llet E and E_n denote the spectral projections of $\mathcal{K}$ and $\mathcal{K}_n$, respectively, associated with their spectra enclosed by a simple closed contour Γ. Let λ be an eigenvalue of $\mathcal{K}$ with algebraic multiplicity m, and let $\hat{\lambda}_n$ denote the arithmetic mean of the m eigenvalues of $\mathcal{K}_n$ inside Γ. Denote the corresponding spectral subspaces by $\mathcal{R}(E)$ and $\mathcal{R}(E_n)$. For any $\psi_n \in \mathcal{R}(E_n)$, define $\psi_n^S = \mathcal{K}\psi_n$. Then, for all sufficiently large n, there exists a constant $C > 0$, independent of n, such that*

$$\|\psi_n - E\psi_n\| \leq \hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n)\big) \leq C\big\|(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|,$$

$$\big\|\psi_n^S - E\psi_n^S\big\| \leq \delta\big(\mathcal{R}(E), \mathcal{K}\mathcal{R}(E_n)\big) \leq C\big\|\mathcal{K}(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|,$$

and

$$|\lambda - \hat{\lambda}_n| \leq C\big\|\mathcal{K}(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|.$$

Since $\mathcal{K}_n^M$ converges to $\mathcal{K}$ in the operator norm, it follows that for all sufficiently large n, the operator $\mathcal{K}_n^M$ has exactly m eigenvalues $\lambda_{n,1}^M, \lambda_{n,2}^M, \ldots, \lambda_{n,m}^M$ inside the contour Γ, counted according to algebraic multiplicity. Let

$$\hat{\lambda}_n^M = \frac{1}{m}\sum_{j=1}^{m} \lambda_{n,j}^M$$

denote the arithmetic mean of these eigenvalues, and let E_n^M denote the corresponding spectral projection. Then, as a consequence of Lemmas 1.5.2 and 1.5.3, the following result holds.

Lemma 1.5.4 (Kulkarni [52]). *For all sufficiently large n, there exists a constant $C > 0$, independent of n, such that*

$$\hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n^M)\big) \leq C \left\| (\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)} \right\|,$$

and

$$\left|\lambda - \hat{\lambda}_n^M\right| \leq C \left\| \mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)} \right\|.$$

The following result can be obtained from the above Lemma 1.5.4.

Theorem 13. *Let $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ converging to $\mathcal{K}$ in the collectively compact sense, with $\mathcal{K}_n = \mathcal{K}_n^M$. Let E and E_n^M denote the spectral projections of $\mathcal{K}$ and $\mathcal{K}_n^M$, respectively, associated with their spectra enclosed by a simple closed contour Γ. Let λ be an eigenvalue of $\mathcal{K}$ with algebraic multiplicity m, and let $\hat{\lambda}_n^M$ denote the arithmetic mean of the m eigenvalues of $\mathcal{K}_n^M$ inside Γ. Denote the corresponding spectral subspaces by $\mathcal{R}(E)$ and $\mathcal{R}(E_n^M)$. For any $\psi_n^M \in \mathcal{R}(E_n^M)$, $\widetilde{\psi}_n^M = \mathcal{K}\psi_n^M$. Then, for all sufficiently large n, there exists a constant $C > 0$, independent of n, such that*

$$\left\|\psi_n^M - E\psi_n^M\right\| \leq \hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n^M)\big) \leq C\left\|(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\|,$$

$$\left\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\right\| \leq \delta\big(\mathcal{R}(E), \mathcal{K}\mathcal{R}(E_n^M)\big) \leq C\left\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\|,$$

and

$$|\lambda - \hat{\lambda}_n^M| \leq C\left\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\|.$$

Theorems 12 and 13 form the core analytical results of this section. These results can also be found in Bouda et al. [16]. They provide rigorous and quantitative error estimates for both eigenfunctions and eigenvalues arising from projection-based and modified projection-based approximations of the compact operator $\mathcal{K}$.

Specifically, Theorem 12 establishes the convergence of the approximate eigenspaces $\mathcal{R}(E_n)$ to the exact eigenspace $\mathcal{R}(E)$ and provides error bounds for the corresponding eigenfunctions and for the arithmetic mean of the approximate eigenvalues. Also, Theorem 13 extends these results to modified projection methods and shows that improved error bounds are obtained in this case, indicating the enhanced accuracy of the modified methods. Together, these theorems provide the theoretical foundation for the convergence and improved convergence behaviour of the numerical methods analyzed in the Chapter 4.

Consider a uniform partition of $[0, 1]$ given by $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$, where $t_j = \dfrac{j}{n}$ for $j = 0, 1, \ldots, n$, and let $h = \dfrac{1}{n}$. For $r \geq 0$, let $\mathcal{X}_n$ denote the space of piecewise polynomials of even degree $\leq 2r$ associated with this partition. In Chapter 4, we investigate the eigenvalue problem $\mathcal{K}\psi = \lambda\psi$, where $\mathcal{K}$ is a compact linear operator. The analysis focuses on the convergence rates of eigenvalue and spectral subspace approximations for integral operators with smooth kernels as well as for operators with Green's function-type kernels. Using orthogonal and interpolatory projection methods at $2r + 1$ equidistant collocation points (not necessarily Gauss points) onto the space $\mathcal{X}_n$ of even-degree piecewise polynomials, we study the convergence behaviour of standard, iterated, and modified projection methods. Superconvergence rates for eigenvalue approximations obtained through the modified projection method are established. In particular, we established that the iterated and modified projection approaches yield improved convergence rates for the associated eigenfunctions compared to the corresponding standard projection methods. The theoretical results are rigorously justified and further supported by numerical experiments.

Unlike earlier works [49, 52], which address smooth kernels using orthogonal projection and Gauss-point-based interpolatory projections, the present study primarily focuses on integral operators with Green's function-type kernels and employs orthogonal as well as interpolatory projections based on equidistant collocation points. Extending the superconvergence results developed in Chapter 2 for linear Fredholm integral equations to the corresponding eigenvalue problems, we prove that iteration preserves the superconvergence of eigenfunctions, while the modified projection method enhances the convergence rate of eigenvalue approximations, thereby demonstrating the efficiency and flexibility of the proposed framework.

1.6 Organisation of Thesis

The thesis is organised as follows:

Chapter 2 presents an analytical study of projection-based numerical methods for the linear equation $x - \mathcal{K}x = f$, where $\mathcal{K}$ is a Fredholm integral operator on $\mathcal{C}[0,1]$. The analysis covers both smooth kernels and kernels of Green's function-type. For $r \geq 0$, using interpolatory projections at $2r+1$ equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree $\leq 2r$, we analyze the convergence of the standard, iterated, modified, and iterated modified collocation methods. For smooth kernels, the modified methods yield higher-order convergence and superconvergence, while for Green's function-type kernels they retain their superior accuracy despite reduced regularity. A single iteration step is also shown to enhance the accuracy of both standard and modified methods. Thus, the results hold even with simple equidistant collocation points, highlighting the robustness of our projection framework. Numerical experiments are given to support the theoretical findings.

Chapter 3 extends the analysis to the nonlinear integral equation $x - \mathcal{K}(x) = f$, where $\mathcal{K}$ is a Urysohn integral operator acting on $\mathcal{L}^\infty[0,1]$ and has either a smooth kernel or a Green's function-type kernel. By employing the same interpolatory projection at equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree, we establish that the iterated methods achieve accelerated convergence over both the standard and modified collocation methods. The order of convergence of each method aligns well with the results obtained for the linear integral equation discussed in the previous chapter, for both smooth kernels and Green's function-type kernels. Numerical experiments are provided to validate and illustrate the theoretical findings.

Chapter 4 examines the eigenvalue problem $\mathcal{K}\psi = \lambda\psi$, for compact Fredholm integral operator $\mathcal{K}$ acting on $\mathcal{C}[0,1]$ or $\mathcal{L}^2[0,1]$. The analysis addresses two classes of kernels: smooth kernels and kernels of Green's function type. Using orthogonal and interpolatory projections at equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree, we derive convergence rates for the approximation of simple eigenvalues and the corresponding spectral subspaces. The modified projection method is shown to be particularly effective, providing significantly higher convergence rates for both eigenvalue and eigenfunction approximations. Furthermore, the iterated projection methods yield superconvergent eigenfunction approximations. These advantages persist for both smooth and Green's function-type kernels. Numerical results included at the end of the chapter confirm the theoretical findings.

Finally, **Chapter 5** presents a summary of the principal findings of the thesis along with limitations and outlines possible directions for future research.

Chapter 2

Fredholm Integral Equations with Smooth and Green's Kernels[1]

This chapter presents an analytical study of projection-based numerical methods for solving Fredholm integral equations of the second kind on the space of continuous functions over the interval $[0, 1]$. The approach is based on finite-dimensional subspaces of piecewise polynomials of even degree, defined over a uniform partition, together with an interpolatory projection operator constructed from equally spaced collocation points. Within this framework, four numerical methods are considered: the standard collocation, iterated collocation, modified collocation, and iterated modified collocation methods. For smooth kernels, the standard and iterated collocation methods exhibit moderate convergence, while the modified and iterated modified methods achieve substantially higher accuracy, demonstrating superconvergence. The study is further extended to equations with non-smooth kernels of the Green's function-type, where the modified methods retain their superior convergence behaviour despite kernel singularities. Moreover, a single iteration step is shown to enhance the accuracy of both standard and modified collocation solutions. The results highlight that these improved convergence rates are obtained even with simple equidistant (non-Gaussian) collocation points, confirming the robustness and efficiency of the proposed projection framework. Numerical experiments presented at the end validate the theoretical findings.

[1] A portion of the work presented in this chapter has been published in *The Journal of Analysis*, vol. 33, pp. 1599-1621, 2025.

2.1 Preliminaries

Consider the Banach space $\mathcal{X} = \mathcal{C}[0,1]$ equipped with the supremum norm. Suppose a function κ is continuous real-valued function on $\Omega = [0,1] \times [0,1]$, and define the integral operator

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\, dt, \quad s \in [0,1], \quad x \in \mathcal{X}. \tag{2.1.1}$$

The operator $\mathcal{K}$ is bounded linear on the space $\mathcal{X}$. The operator equation under consideration is

$$x - \mathcal{K}x = f, \quad f \in \mathcal{X}. \tag{2.1.2}$$

Since κ is continuous on the compact domain Ω, the operator $\mathcal{K}$ is compact on $\mathcal{X}$. The function f is assumed to be given. Assume that 1 is not in the spectrum of $\mathcal{K}$, ensuring that $(\mathcal{I}-\mathcal{K})^{-1}$ exists and must be bounded. The equation (2.1.2) admits a unique solution, denoted by φ. Moreover, if $f \in \mathcal{C}^{\alpha}[0,1]$ for $\alpha \geq 0$, it follows that $\varphi \in \mathcal{C}^{\alpha}[0,1]$ as well; see [21]. Since finding the exact solution is challenging, a class of numerical methods is employed to approximate the solution. These methods include the classical Galerkin and collocation approaches, which rely on projecting operators onto subspaces. In these approaches, the sequence of projections converges pointwise to the identity operator, ensuring the accuracy of the approximation. In this chapter, we will discuss the collocation methods as well as its variants, analyzing the orders of convergence for both smooth and non-smooth kernels of Green's function-type.

Let $\mathcal{X}_n$ denote a finite-dimensional space, consisting of piecewise polynomial functions defined over a uniform partition of the interval $[0,1]$ into n subintervals, each of length $h = \dfrac{1}{n}$. Within each subinterval, a fixed number of distinct interpolation points are chosen, referred to as collocation points. Let $\mathcal{Q}_n : \mathcal{X} \to \mathcal{X}_n$ denote the corresponding interpolatory operator, which maps a continuous function onto its interpolant in $\mathcal{X}_n$ determined by the prescribed interpolation nodes. For any $\varphi \in \mathcal{C}[0,1]$, the projection satisfies $\mathcal{Q}_n\varphi \to \varphi \quad \text{as} \quad n \to \infty$, which follows from the uniform convergence of interpolants on the refined partitions. By the Hahn-Banach theorem, $\mathcal{Q}_n$ can be extended continuously to $\mathcal{L}^{\infty}[0,1]$, thereby defining a bounded projection operator $\mathcal{Q}_n : \mathcal{L}^{\infty}[0,1] \to \mathcal{X}_n$. We now proceed to review the construction of the collocation method and its modified variants based on this general interpolatory framework.

In the collocation method, the original equation (2.1.2) is approximated by

$$\varphi_n^C - \mathcal{Q}_n\mathcal{K}\varphi_n^C = \mathcal{Q}_n f. \tag{2.1.3}$$

The corresponding iterated collocation solution is obtained by a single iteration step as

$$\varphi_n^S = \mathcal{K}\varphi_n^C + f. \tag{2.1.4}$$

The modified collocation method is defined by

$$\varphi_n^M - \mathcal{K}_n^M \varphi_n^M = f, \tag{2.1.5}$$

where the modified operator $\mathcal{K}_n^M$ is given by

$$\mathcal{K}_n^M = \mathcal{Q}_n\mathcal{K} + \mathcal{K}\mathcal{Q}_n - \mathcal{Q}_n\mathcal{K}\mathcal{Q}_n.$$

An iterated version of the modified collocation method is then defined as

$$\widetilde{\varphi}_n^M = \mathcal{K}\varphi_n^M + f, \tag{2.1.6}$$

which yields the iterated modified collocation solution.

2.1.1 Interpolatory Projection

We briefly recall the interpolatory projection operator $\mathcal{Q}_n$, as introduced in Chapter 1. Let the interval $[0, 1]$ be uniformly partitioned into n subintervals

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n,$$

with mesh size $h = \dfrac{1}{n}$ and subintervals $\Delta_j = [t_{j-1}, t_j]$, $j = 1, 2, \ldots, n$.

For $r \geq 0$ be an integer and we consider the piecewise polynomial space

$$\mathcal{X}_n = \Big\{ x \in \mathcal{L}^\infty[0, 1] : x|_{\Delta_j} \text{ is a polynomial of even degree } \leq 2r, j = 1, 2, \ldots, n \Big\}$$

as an approximating space with respect to the above partition.

For $r \neq 0$, the interpolation nodes in Δ_j are chosen as

$$\tau_j^i = t_{j-1} + \frac{ih}{2r} = t_{j-1} + h\zeta_i, \quad \zeta_i = \frac{i}{2r}, \quad i = 0, 1, \ldots, 2r.$$

For $r = 0$, a single midpoint node is used as

$$\tau_j = \frac{t_{j-1} + t_j}{2}.$$

The interpolatory operator $\mathcal{Q}_n : \mathcal{X} \to \mathcal{X}_n$ is defined such that for each subinterval Δ_j of the uniform partition,

$$\mathcal{Q}_n x(\tau_j^i) = x(\tau_j^i), \quad i = 0, 1, \ldots, 2r, \tag{2.1.7}$$

where τ_j^i are the prescribed interpolation nodes in Δ_j. It follows that

$$\mathcal{Q}_n x \to x \quad \text{for all } x \in \mathcal{C}[0,1],$$

as the mesh size $h \to 0$. The interpolatory operator $\mathcal{Q}_n$ can be extended to a bounded linear operator on $\mathcal{L}^\infty[0,1]$ by extending the associated bounded linear functionals using the Hahn-Banach theorem.

Let $x \in \mathcal{C}[0,1]$, and let $\{\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\}$ be distinct interpolation points in the subinterval $\Delta_j = [t_{j-1}, t_j]$. Denote the first divided difference of x at τ_j^0 by $\left[\tau_j^0\right] x = x(\tau_j^0)$, and the $(2r+1)^{\text{th}}$ divided difference of x at the points $\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}$ by

$$\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\right] x = \frac{\left[\tau_j^1, \tau_j^2, \ldots, \tau_j^{2r}\right] x - \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r-1}\right] x}{\tau_j^{2r} - \tau_j^0},$$

and so on recursively.

Let $\Psi_j(t)$ denote the interpolation node polynomial over Δ_j, defined as

$$\Psi_j(t) = \prod_{i=0}^{2r} (t - \tau_j^i).$$

Since $\mathcal{Q}_{n,j} x$ (where $\mathcal{Q}_{n,j}$ denotes the restriction of $\mathcal{Q}_n$ to Δ_j) is the polynomial that interpolates x at the points $\{\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\}$, the classical interpolation error formula yields

$$x(t) - \mathcal{Q}_{n,j} x(t) = \Psi_j(t) \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t\right] x, \quad t \in \Delta_j.$$

As established in Chapter 1, if $x \in \mathcal{C}^{2r+1}(\Delta_j)$, the interpolation error satisfies the classical estimate

$$\|(\mathcal{I} - \mathcal{Q}_{n,j}) x\|_{\Delta_j,\infty} \leq C_1 \left\| x^{(2r+1)} \right\|_{\Delta_j,\infty} h^{2r+1},$$

where C_1 is a constant independent of h. Consequently, for any $x \in \mathcal{C}^{2r+1}[0,1]$, the global error satisfies

$$\| (\mathcal{I} - \mathcal{Q}_n)\, x \|_\infty \leq C_1 \left\| x^{(2r+1)} \right\|_\infty h^{2r+1}. \tag{2.1.8}$$

The analysis presented in this chapter is devoted to establishing higher-order convergence results for the iterated versions of the collocation and modified collocation methods in the case of smooth kernels, and to extending this analysis to integral equations with Green's function-type kernels. The chapter has been arranged in the following way. In Section 2.2, we present the convergence analysis for integral equations involving smooth kernels. Section 2.3 is devoted to the convergence behaviour corresponding to kernels of Green's function-type. The numerical experiments supporting the theoretical results are reported in Section 2.4, while concluding remarks are summarized in Section 2.5.

2.2 Convergence Analysis for Smooth Kernels

In this section, we consider the Fredholm integral operator $\mathcal{K}$ defined by (2.1.1) with a smooth kernel $\kappa \in \mathcal{C}^{2r+2}(\Omega)$, given by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1], \quad x \in \mathcal{X}.$$

Then $\mathcal{K} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is a compact linear operator. We further assume that the integral operator $\mathcal{K}$ admits a continuous extension to $\mathcal{L}^\infty[0,1]$ and satisfies $\mathcal{R}(\mathcal{K}) \subset \mathcal{C}^{2r+2}[0,1]$. This assumption is satisfied for smooth kernels and is required for the subsequent error analysis. The corresponding Fredholm integral equation of the second kind can be expressed as

$$x - \mathcal{K}x = f, \quad f \in \mathcal{X}.$$

As in Section 2.1, we assume that the above equation admits a unique solution φ. If $f \in \mathcal{C}^{2r+1}[0,1]$, then the solution satisfies $\varphi \in \mathcal{C}^{2r+1}[0,1]$. Moreover, it is known that the collocation approximation φ_n^C satisfies the following error estimate:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,.$$

When the collocation points of the interpolatory projection $\mathcal{Q}_n$ are chosen arbitrarily, either as partition points or any other points that are not the roots of special orthogonal polynomials, the iterated collocation solution φ_n^S may fail to exhibit superconvergence. This behaviour can be demonstrated by the following example.

Example 1. *Let $\mathcal{X} = \mathcal{C}[0,1]$, and let $\mathcal{X}_n$ be the space of piecewise constant functions with respect to the uniform partition of* $[0,1]$ *as*

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0,1,\ldots,n.$$

Define the interpolatory operator $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ by interpolation at points

$$\tau_j = \frac{3j-2}{3n}, \quad j = 1,2,\ldots,n.$$

Consider the compact linear operator $\mathcal{K}$ defined by

$$(\mathcal{K}x)(s) = \int_0^1 s\,x(t)\,dt, \quad s \in [0,1].$$

Since 1 *is not an eigenvalue of $\mathcal{K}$, the Fredholm equation*

$$x - \mathcal{K}x = f, \quad f(s) = 2s, \quad s \in [0,1],$$

has the exact solution $\varphi(s) = 4s$.

Now, for $s \in [t_{j-1}, t_j]$, *we have*

$$(\mathcal{I} - \mathcal{Q}_n)\varphi(s) = 4s - \frac{4(3j-2)}{3n}.$$

Therefore,

$$\int_0^1 (\mathcal{I} - \mathcal{Q}_n)\varphi(t)\,dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(4t - \frac{4(3j-2)}{3n}\right) dt = \frac{2}{3n}.$$

It follows that

$$\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = \max_{s\in[0,1]} \left| s \int_0^1 (\mathcal{I} - \mathcal{Q}_n)\varphi(t)\,dt \right| = \frac{2}{3n},$$

and

$$\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = \max_{1\le j\le n} \left\{ \max_{s\in[t_{j-1},t_j]} |(\mathcal{I} - \mathcal{Q}_n)\varphi(s)| \right\} \le \frac{8}{3n}.$$

By a result in [13], the iterated collocation solution is superconvergent if and only if

$$\frac{\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty}{\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty} \to 0 \quad \text{as } n \to \infty.$$

In this case, the condition fails since

$$\frac{\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty}{\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty} \ge \frac{1}{4}.$$

Hence, no improvement in the convergence rate is achieved through iteration. Indeed, from [48, Theorem 3.1], for $\kappa \in \mathcal{C}^{2r+1}[0,1]$, *we have*

$$\left\|\varphi - \varphi_n^M\right\|_\infty = \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^{4r+2}\right).$$

Superconvergence under Appropriate Interpolation methods

An appropriate selection of interpolation points can significantly enhance the convergence rate of both the collocation and the modified collocation methods. Recall the uniform partition of the interval $[0,1]$ given as

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad h = \frac{1}{n}.$$

Within each subinterval $[t_{j-1}, t_j]$, we define $2r+1$ interpolation points as

$$\tau_j^i = t_{j-1} + \zeta_i h, \quad i = 0, 1, 2, \ldots, 2r,$$

where $\{\zeta_i\}_{i=0}^{2r}$ are distinct points in $[0,1]$.

For a non-negative integer r, let the finite-dimensional approximating space $\mathcal{X}_n$ be the space of piecewise polynomial functions on $[0,1]$, each of degree not exceeding $2r$ on every subinterval of the given uniform partition.

(i) Gauss Interpolation Points: If $\{\zeta_0, \zeta_1, \dots, \zeta_{2r}\}$ are chosen as the Gauss points (i.e., the roots of the Legendre polynomial of degree $\leq 2r$) on $[0,1]$, and if $\kappa \in \mathcal{C}^{4r+2}(\Omega)$ and $f \in \mathcal{C}^{4r+2}[0,1]$, then from Atkinson [13, Chapter 3], the following error estimates hold:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{4r+2}\big)\,.$$

Furthermore, Kulkarni [48, Theorem 4.5] established that

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{6r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{8r+4}\big)\,.$$

(ii) Equidistant Interpolation Points: Alternatively, if the interpolation points are chosen as equidistant nodes within each subinterval, i.e.,

$$\zeta_i = \frac{i}{2r}, \quad i = 0, 1, 2, \dots, 2r,$$

and if $\kappa \in \mathcal{C}^{4r+2}(\Omega)$ and $\varphi \in \mathcal{C}^{4r}[0,1]$, then following Atkinson [13, Section 3.4.3], we obtain

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+2}\big)\,.$$

In this context, Rane [75] investigated the modified projection method and its iterated version for the space of piecewise quadratic polynomials defined over the same partition of $[0,1]$. The following convergence results, stated therein (without proof), hold for the approximating space $\mathcal{X}_n$:

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{4r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{4r+4}\big)\,.$$

A rigorous proof of these estimates will be provided in a subsequent section, utilizing fundamental results from elementary analysis.

In both of the aforementioned cases, it is evident that a single iteration of the projection-based approximation substantially accelerates the convergence rate, demonstrating the critical role of interpolation point selection in achieving superconvergence.

2.2.1 Error Estimates for Smooth Kernels

In this section, we introduce the notation and several preliminary results that will be utilized in subsequent convergence analysis. Let α and β be non-negative integers such that $0 \leq \alpha + \beta \leq 2r + 3$. We define the mixed partial derivative of the kernel κ by

$$D^{(\alpha,\beta)}\kappa(s,t) = \frac{\partial^{\alpha+\beta}\kappa}{\partial s^\alpha \partial t^\beta}(s,t),$$

and set
$$C_2 = \max_{0\le\alpha+\beta\le 2r+3}\left\{\sup_{0\le s,t\le 1}\left|D^{(\alpha,\beta)}\kappa(s,t)\right|\right\}.$$
Here, the constant C_2 provides a uniform upper bound for all partial derivatives of κ up to total order $2r+3$.

For any non-negative integer γ, if $x \in \mathcal{C}^{2r+\gamma}[0,1]$, we define the corresponding maximum norm as
$$\|x\|_{2r+\gamma,\infty} = \max_{0\le i\le 2r+\gamma}\left\|x^{(i)}\right\|_\infty = \max_{0\le i\le 2r+\gamma}\left\{\sup_{t\in[0,1]}\left|x^{(i)}(t)\right|\right\},$$
where $x^{(i)}$ denotes the i-th derivative of x. In particular, the usual supremum norm on $\mathcal{C}[0,1]$ is denoted by
$$\|x\|_\infty = \sup_{t\in[0,1]}|x(t)|.$$

We now establish several propositions that quantify the error behaviour of integral operators involving $\mathcal{K}$ and $\mathcal{Q}_n$, under the assumption that the kernel κ is smooth.

Proposition 14. *If the kernel κ is continuously differentiable with respect to t and $x \in \mathcal{C}^{2r+2}[0,1]$, then*
$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_2\|x\|_{2r+2,\infty}h^{2r+2},$$
where C_2 is a constant independent of h.

Proof. For $s \in [0,1]$, we write
$$\begin{aligned}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) &= \int_0^1 \kappa(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt\\ &= \sum_{j=1}^n\int_{t_{j-1}}^{t_j}\kappa(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})x(t)\,dt.\end{aligned}$$
For each subinterval $[t_{j-1},t_j]$, the interpolation remainder can be expressed as
$$(\mathcal{I}-\mathcal{Q}_{n,j})x(t) = \Psi_j(t)\,[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]x, \quad t\in[t_{j-1},t_j],$$
where $\Psi_j(t) = \prod_{i=0}^{2r}\big(t-\tau_j^i\big)$. Then,
$$\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{j=1}^n\int_{t_{j-1}}^{t_j}\kappa(s,t)\Psi_j(t)\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]x\,dt,$$
For fixed s, let
$$f_j^s(t) = \kappa(s,t)\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]x, \quad t\in[t_{j-1},t_j], \text{ for } j=1,2,\ldots,n.$$

Then

$$\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f_j^s(t)\Psi_j(t)\,dt \tag{2.2.1}$$

For $j = 1, 2, \ldots, n$, define

$$\omega_j(t) = \int_{t_{j-1}}^{t} \Psi_j(s)\,ds.$$

Then, by Leibnitz rule

$$\omega_j'(t) = \Psi_j(t), \ \ \omega_j(t) \geq 0 \ \ \text{for } t \in [t_{j-1}, t_j],$$

Obseve that, $\omega_j(t_{j-1}) = 0$ and by Lemma 1.3.1 $\omega_j(t_j) = 0$.
Using above result in equation (2.2.1), we have

$$\begin{aligned}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) &= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f_j^s(t)\omega_j'(t)\,dt \\ &= -\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (f_j^s)'(t)\omega_j(t)\,dt. \end{aligned} \tag{2.2.2}$$

Note that,

$$(f_j^s)'(t) \ = \ \kappa(s,t)\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t, t\right]x + D^{(0,1)}\kappa(s,t)\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t\right]x.$$

From Hilderbrand [36], we recall standard result of divided difference as follows:
Since $x \in \mathcal{C}^{2r+2}[0,1]$ and $t \in [t_{j-1}, t_j]$, we have

$$\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t\right]x = \frac{x^{(2r+1)}(\xi_1^j)}{(2r+1)!}, \quad \text{and} \quad \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t, t\right]x = \frac{x^{(2r+2)}(\xi_2^j)}{(2r+2)!},$$

for some $\xi_1^j, \xi_2^j \in [t_{j-1}, t_j]$. Therefore, for $s \in [0,1]$ and $t \in [t_{j-1}, t_j]$,

$$\left|(f_j^s)'(t)\right| \leq 2C_2\|x\|_{2r+2,\infty}.$$

Also, for $j = 1, 2, \ldots, n$, we observe that

$$\|\Psi_j\|_\infty = \max_{t\in[0,1]} |\Psi_j(t)| = O\big(h^{2r+1}\big).$$

Using (2.2.2), we obtain

$$\begin{aligned}|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s)| &\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} |(f_j^s)'(t)||\Psi_j(s)|\,ds\,dt \\ &\leq 2C_2\|x\|_{2r+2,\infty}h^{2r+2}.\end{aligned}$$

Thus, we have

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq 2C_2\|x\|_{2r+2,\infty}h^{2r+2},$$

which completes the proof. □

Proposition 15. *If the kernel $\kappa \in \mathcal{C}^{2r+1}(\Omega)$ and $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\big(h^{4r+3}\big).$$

Proof. Let $\kappa \in \mathcal{C}^{2r+1}(\Omega)$ and let $x \in \mathcal{C}^{2r+2}[0,1]$. Then $\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x \in \mathcal{C}^{2r+1}[0,1]$. Using the interpolation error estimate (2.1.8), we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq C_1\big\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)^{(2r+1)}\big\|_\infty h^{2r+1}. \tag{2.2.3}$$

Consider

$$\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) = \int_0^1 \kappa(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt.$$

Since $\kappa \in \mathcal{C}^{2r+1}(\Omega)$, differentiation under the integral sign is valid, yielding

$$\begin{aligned}
\left(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)^{(2r+1)}(s) &= \int_0^1 \frac{\partial^{2r+1}\kappa}{\partial s^{2r+1}}(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} D^{(2r+1,0)}\kappa(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})x(t)\,dt \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} D^{(2r+1,0)}\kappa(s,t)\Psi_j(t)[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]x\,dt.
\end{aligned}$$

For fixed s, we assume that

$$g_j^s(t) = D^{(2r+1,0)}\kappa(s,t)\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]x, \quad t\in[t_{j-1},t_j],\ \text{for } j=1,2,\ldots,n.$$

Therefore,

$$\left(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)^{(2r+1)}(s) = \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\,dt$$

It is easy to see that, for $s\in[0,1]$ and $t\in[t_{j-1},t_j]$,

$$\left|(g_j^s)'(t)\right| \leq 2C_2\|x\|_{2r+2,\infty}.$$

Let c_j be the midpoint of Δ_j. By the mean-value form of Taylor's theorem, for each $t\in\Delta_j$ there exists $\xi_t\in\Delta_j$ such that

$$g_j^s(t) = g_j^s(c_j) + (t-c_j)(g_j^s)'(\xi_t).$$

Using the antisymmetry of Ψ_j about c_j (Lemma 1.3.1, Chapter 1), we have $\int_{t_{j-1}}^{t_j}\Psi_j(t)\,dt = 0$, and hence

$$\int_{t_{j-1}}^{t_j} g_j^s(t)\,\Psi_j(t)\,dt = \int_{t_{j-1}}^{t_j}(t-c_j)(g_j^s)'(\xi_t)\,\Psi_j(t)\,dt.$$

Taking absolute values, we obtain

$$\left|\int_{t_{j-1}}^{t_j} g_j^s(t)\,\Psi_j(t)\,dt\right| \le 2C_2\|x\|_{2r+2,\infty}\int_{t_{j-1}}^{t_j} |t-c_j|\,|\Psi_j(t)|\,dt.$$

On Δ_j we have $|t-c_j|\le h/2$ and, since each factor $(t-\tau_j^i)=O(h)$ on Δ_j,

$$\int_{t_{j-1}}^{t_j} |\Psi_j(t)|\,dt = O\big(h^{2r+2}\big)\,.$$

Therefore each subinterval contributes

$$\left|\int_{t_{j-1}}^{t_j} g_j^s(t)\,\Psi_j(t)\,dt\right| = O(h)\cdot O\big(h^{2r+2}\big) = O\big(h^{2r+3}\big)\,,$$

uniformly in s and j. Summing over $j=1,\ldots,n$ $\left(\text{with } n=\dfrac{1}{h}\right)$ yields

$$\big|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)^{(2r+1)}(s)\big| \le n\cdot O\big(h^{2r+3}\big) = \frac{1}{h}\,O\big(h^{2r+3}\big) = O\big(h^{2r+2}\big)\,,$$

uniformly in s. Hence there exists $C_3>0$, independent to h, such that

$$\big\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)^{(2r+1)}\big\|_\infty \le C_3\|x\|_{2r+2,\infty}\,h^{2r+2}.$$

Finally, substituting this into (2.2.3) gives

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le C_1C_3\|x\|_{2r+2,\infty}\,h^{(2r+1)+(2r+2)} = O\big(h^{4r+3}\big)\,,$$

which completes the proof. □

Proposition 16. *If the kernel $\kappa\in\mathcal{C}^{2r+2}(\Omega)$ and $x\in\mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\big(h^{4r+4}\big)\,.$$

Proof. Assume that $\kappa\in\mathcal{C}^{2r+2}(\Omega)$ and let $x\in\mathcal{C}^{2r+2}[0,1]$. Then $\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\in\mathcal{C}^{2r+2}[0,1]$. Using the Proposition 14, we get

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_2\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty}h^{2r+2}. \tag{2.2.4}$$

To estimate the derivative norm $\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty}$, note that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty} = \max_{0\le i\le 2r+2}\left\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)^{(2r+2)}\right\|_\infty.$$

Since $\kappa\in\mathcal{C}^{2r+2}(\Omega)$, and the argument used in Proposition 15 applies. Following the same reasoning, one obtains

$$\left|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)^{(2r+2)}(s)\right| \le 2C_2\|x\|_{2r+2,\infty}h^{2r+2}.$$

From estimate (2.2.4), we have

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 4(C_2)^2\|x\|_{2r+2,\infty}h^{4r+4},$$

which proves the result. □

2.2.2 Collocation Methods and Their Variants

In this subsection, we analyze the convergence properties of the collocation method and its iterated and modified variants for solving the Fredholm integral equation of the second kind, as given in (2.1.2). The discussion builds upon the interpolatory projection operator $\mathcal{Q}_n$ introduced earlier, which is defined over a uniform partition of the interval $[0, 1]$ using piecewise polynomials of even degree $\leq 2r$. The choice of an even-degree polynomial basis ensures the symmetry of interpolation and facilitates higher-order accuracy in the resulting numerical approximations.

Convergence of Collocation and Iterated Collocation Solutions

We begin by recalling that the collocation approximation φ_n^C to the true solution φ is obtained by enforcing the integral equation (2.1.3) at $2r + 1$ collocation points. The corresponding iterated collocation solution φ_n^S is then constructed by applying the integral operator $\mathcal{K}$ once more to improve the accuracy of the collocation solution, as defined in (2.1.4).

According to the error bounds (1.3.7) and (1.3.9) given in Chapter 1, the collocation and iterated collocation errors satisfy the following estimates:

$$\left\|\varphi - \varphi_n^C\right\|_\infty \leq C_4\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty \tag{2.2.5}$$

and

$$\left\|\varphi - \varphi_n^S\right\|_\infty \leq C_5\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty, \tag{2.2.6}$$

where $C_4 = \left\|(\mathcal{I} - \mathcal{Q}_n\mathcal{K})^{-1}\right\| < \infty$ and $C_5 = \left\|(\mathcal{I} - \mathcal{K}\mathcal{Q}_n)^{-1}\right\| < \infty$ is a constant independent of h.

By applying inequality (2.1.8) and Proposition 14, we obtain the following error bounds:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\left(h^{2r+1}\right), \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\left(h^{2r+2}\right).$$

This clearly demonstrates that the one-step iteration improves the convergence order by one, validating the use of the iterated collocation method as a refinement over the standard collocation method.

Convergence of Modified Collocation Solution and Its Iterated Form

We now turn to an enhanced version of the collocation method, the modified collocation method, which introduces a correction term involving the projection residual

$(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)$. This modification significantly improves the approximation properties, especially when the kernel κ is smooth. The convergence behaviour of both the modified collocation solution φ_n^M and the iterated modified collocation solution $\widetilde{\varphi}_n^M$ is summarized by the following theorem.

Theorem 17. *Let $\mathcal{K}$ be the Fredholm integral operator defined by* (2.1.1) *with a smooth kernel $\kappa \in \mathcal{C}^{2r+2}(\Omega)$, and let φ denote the unique solution of* (2.1.2) *for $f \in \mathcal{C}^{2r+2}[0,1]$. Let $\mathcal{Q}_n$ be the interpolation operator onto the subspace $\mathcal{X}_n$ defined by* (2.1.7). *Denote by φ_n^M and $\widetilde{\varphi}_n^M$ the modified collocation and iterated modified collocation solutions, defined respectively by* (2.1.5) *and* (2.1.6). *Then the following error estimates hold:*

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{4r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{4r+4}\big)\,.$$

Proof. To derive these results, we first note that the modified collocation error satisfies

$$\left\|\varphi - \varphi_n^M\right\|_\infty \le C_6\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty, \tag{2.2.7}$$

and the iterated modified collocation error satisfies

$$\begin{aligned}\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty \le& \left\|(\mathcal{I} - \mathcal{K})^{-1}\right\|\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty \\ &+ \left\|(\mathcal{I} - \mathcal{K})^{-1}\right\|\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\|\left\|\varphi - \varphi_n^M\right\|_\infty,\end{aligned} \tag{2.2.8}$$

where C_6 is a constant independent of n; see (1.3.13) and (1.3.14) given in Chapter 1. Using Proposition 15 in (2.2.7), we obtain

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{4r+3}\big)\,.$$

To further estimate the term $\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\|$, observe that $\mathcal{Q}_n$ represents a sequence of projections converging pointwise to the identity operator, and that $\|\mathcal{Q}_n\|$ remains uniformly bounded. Since $\mathcal{K}$ is a bounded linear operator, the problem reduces to finding an upper bound for $\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}\|$. For a smooth kernel κ, inequality (2.1.8) yields

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}x\|_\infty \le C_1\left\|(\mathcal{K}x)^{(2r+1)}\right\|_\infty h^{2r+1} \le C_2C_1\|x\|_\infty h^{2r+1}.$$

Using this estimate, Proposition 16, and the previously obtained bound on $\left\|\varphi - \varphi_n^M\right\|_\infty$ in (2.2.8), we obtain

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{4r+4}\big)\,,$$

thereby completing the proof. $\square$

The results presented in this section confirm that both iteration and modification significantly enhance the accuracy of collocation-based numerical methods for Fredholm integral equations of the second kind. Specifically, the one-step iteration increases the convergence order by one, while the modified collocation framework achieves superconvergence of higher order, depending on the smoothness of the kernel and the degree of the polynomial approximation. These findings demonstrate that, even when collocation points are equidistant rather than Gaussian, the proposed projection framework retains its high efficiency. In subsequent section, we extend this analysis to integral equations with non-smooth Green's function-type kernels, where similar iterative strategies continue to yield accelerated convergence.

2.3 Convergence Analysis for Green's Kernels

Let $\mathcal{X} = \mathcal{C}[0,1]$ and the compact linear integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is defined as

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where $\kappa(s,t)$ is a real-valued continuous kernel function of the Green's function-type. For $r \geq 0$, we assume that the smoothness parameter α satisfies $\alpha > r$. Consider the subsets of Ω as

$$\Omega_1 = \{(s,t) \mid 0 \leq t \leq s \leq 1\}, \quad \Omega_2 = \{(s,t) \mid 0 \leq s \leq t \leq 1\}.$$

Suppose there exist functions $\kappa_i \in \mathcal{C}^{\alpha}(\Omega_i)$, $i = 1,2$, such that the kernel κ can be written as

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t), & (s,t) \in \Omega_1, \\ \kappa_2(s,t), & (s,t) \in \Omega_2, \end{cases}$$

and $\kappa_1(s,s) = \kappa_2(s,s),$ for all $s \in [0,1]$. A kernel κ satisfying these properties is referred to as a Green's kernel. Such kernels are generally non-smooth, as their regularity is restricted to the separate domains Ω_1 and Ω_2, with only continuity across the diagonal.

2.3.1 Divided Difference Bounds

We use the following notations throughout the article: Let j,k be non-negative integer and we set

$$D^{(j,k)}\kappa_i(s,t) = \frac{\partial^{j+k}\kappa_i}{\partial s^j \partial t^k}(s,t), \quad \text{for } i = 1,2.$$

Define

$$C_7 = \max_{j,k=0,1,2} \left\{ \sup_{(s,t)\in\Omega_i} \left|D^{(j,k)}\kappa_i(s,t)\right| : \text{ for } i = 1,2 \right\},$$

For $l = 1, 2$, we define

$$M_l = \max_{1\le k\le n} \left\{ \sup_{t_{k-1}\le t\le s\le t_k} \left|D^{(l,0)}\kappa_1(s,t)\right|, \sup_{t_{k-1}\le s\le t\le t_k} \left|D^{(l,0)}\kappa_2(s,t)\right| \right\}.$$

Now, for $x \in \mathcal{X}$ and $s \in [0,1]$ we have

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt = \int_0^s \kappa_1(s,t)x(t)\,dt + \int_s^1 \kappa_2(s,t)x(t)\,dt.$$

Using the Leibnitz rule

$$(\mathcal{K}x)'(s) = \int_0^s \frac{\partial\kappa_1}{\partial s}(s,t)x(t)\,dt + \int_s^1 \frac{\partial\kappa_2}{\partial s}(s,t)x(t)\,dt,$$

which implies

$$|(\mathcal{K}x)'(s)| \le \left(\sup_{0\le t\le s\le 1} \left|D^{(1,0)}\kappa_1(s,t)\right|\right) s\|x\|_\infty + \left(\sup_{0\le s\le t\le 1} \left|D^{(1,0)}\kappa_2(s,t)\right|\right) (1-s)\|x\|_\infty.$$

Then,

$$\|(\mathcal{K}x)'\|_\infty \le C_7\|x\|_\infty. \tag{2.3.1}$$

Let $x \in \mathcal{C}[0,1]$, and let $\{\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\}$ be distinct points in in $[0,1]$. For any $s \in [0,1]$, the divided difference of x at the points $\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}$ and s is denoted by $\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right] x$. The key observation is that

$$\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right] \mathcal{K}x = \int_0^1 \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right]\ \kappa(\cdot,t)\,x(t)\,dt,$$

where $\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right]\ \kappa(\cdot,t)$ denotes the divided difference of the kernel function κ with respect to its first variable.

In the following, we prove two important lemmas based on the divided difference of $\mathcal{K}x$ at $\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}$ and s, denoted by $\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right] \mathcal{K}x$.

Lemma 2.3.1. *Let $r \ge 0$ and $\left\{\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\right\}$ be the set of collocation points in the subinterval $[t_{j-1}, t_j]$ for $j = 1, 2, \ldots, n$. Then, for some constant C_{10}, we have*

$$\sup_{s\in[t_{j-1},t_j]} \left|\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right] \mathcal{K}x\right| \le C_{10}\,\|x\|_\infty\, h^{-2r}.$$

Proof. Let the collocation points in each subinterval be given by

$$\tau_j^i = t_{j-1} + \frac{ih}{2r} = t_{j-1} + h\zeta_i, \quad \text{for } i = 0, 1, \ldots, 2r \text{ and } j = 1, 2, \ldots, n,$$

where $\zeta_i = \dfrac{i}{2r} \in [0,1]$. Clearly, $\tau_j^i \in [t_{j-1}, t_j]$.

The integral operator $\mathcal{K}$ is defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt,$$

where the kernel κ is assumed to be Green's function-type, given by

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t), & 0 \le t \le s \le 1, \\ \kappa_2(s,t), & 0 \le s \le t \le 1. \end{cases}$$

Let $s \in [t_{j-1}, t_j]$. Then the divided difference $[\tau_j^0, \tau_j^1, s]\mathcal{K}x$ can be written as

$$\begin{aligned} [\tau_j^0, \tau_j^1, s]\mathcal{K}x &= \int_0^1 [\tau_j^0, \tau_j^1, s]\kappa(\cdot, t)\, x(t)\, dt \\ &= \sum_{\substack{k=1 \\ k\neq j}}^{n} \int_{t_{k-1}}^{t_k} [\tau_j^0, \tau_j^1, s]\kappa(\cdot, t)\, x(t)\, dt + \int_{t_{j-1}}^{t_j} [\tau_j^0, \tau_j^1, s]\kappa(\cdot, t)\, x(t)\, dt. \end{aligned} \tag{2.3.2}$$

Since κ is sufficiently smooth in each interval $[t_{k-1}, t_k]$ for $k \neq j$, we can bound the first summation term as

$$\left| \int_{t_{k-1}}^{t_k} [\tau_j^0, \tau_j^1, s]\kappa(\cdot, t)\, x(t)\, dt \right| \le M_2 \left\| x \right\|_\infty h, \tag{2.3.3}$$

since κ and its partial derivatives up to second order are bounded on $[t_{k-1}, t_k]$, the divided difference $[\tau_j^0, \tau_j^1, s]\kappa(\cdot, t)$ is uniformly bounded by M_2, and the integral over an interval of length h then gives a bound of order $M_2 h\|x\|_\infty$.

Next, we focus on estimating the second term in (2.3.2), corresponding to the local interval $[t_{j-1}, t_j]$. To this end, we derive a bound for $[\tau_j^0, s]\kappa(\cdot, t)$ for $s \in [t_{j-1}, t_j]$.

Case (I): $t_{j-1} \le s < \tau_j^0$. In this situation, the first divided difference of κ with respect to the first variable is given by

$$[\tau_j^0, s]\kappa(\cdot, t) = \frac{\kappa(s,t) - \kappa(\tau_j^0, t)}{s - \tau_j^0}.$$

Depending on the position of t relative to s and τ_j^0, we have

$$[\tau_j^0, s]\kappa(\cdot, t) = \begin{cases} \dfrac{\kappa_1(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0}, & t_{j-1} \le t \le s, \\ \dfrac{\kappa_2(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0}, & s < t < \tau_j^0, \\ \dfrac{\kappa_2(s,t) - \kappa_2(\tau_j^0, t)}{s - \tau_j^0}, & \tau_j^0 \le t \le t_j. \end{cases}$$

For a fixed $s \in [t_{j-1}, \tau_j^0]$, the function $[\tau_j^0, s]\kappa(\cdot, t)$ is continuous on $[t_{j-1}, t_j]$. Since both $\kappa_1(., t)$ and $\kappa_2(., t)$ are continuously differentiable with respect to s, by the Mean Value Theorem there exist points $\eta_1, \eta_2 \in (s, \tau_j^0)$ such that

$$\frac{\kappa_1(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\kappa_1(\eta_1, t), \quad \frac{\kappa_2(s,t) - \kappa_2(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\kappa_2(\eta_2, t).$$

Hence, for $t_{j-1} \le t \le s$, we obtain

$$\left|\frac{\kappa_1(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0}\right| = \left|D^{(1,0)}\kappa_1(\eta_1, t)\right| \le \sup_{t_{j-1} \le t \le s \le t_j} \left|D^{(1,0)}\kappa_1(s, t)\right| = M_1.$$

Similarly, for $\tau_j^0 \le t \le t_j$,

$$\left|\frac{\kappa_2(s,t) - \kappa_2(\tau_j^0, t)}{s - \tau_j^0}\right| = \left|D^{(1,0)}\kappa_2(\eta_2, t)\right| \le \sup_{t_{j-1} \le s \le t \le t_j} \left|D^{(1,0)}\kappa_2(s, t)\right| = M_1.$$

Therefore, we obtain

$$\left|[\tau_j^0, s]\kappa(\cdot, t)\right| \le M_1, \quad \text{for } t_{j-1} \le t \le s \text{ and } \tau_j^0 \le t \le t_j. \tag{2.3.4}$$

Now, consider the intermediate region $s < t < \tau_j^0$. We write

$$\begin{aligned}\frac{\kappa_2(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} &= \frac{\kappa_2(s,t) - \kappa_2(t,t) + \kappa_1(t,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} \\ &= \frac{D^{(1,0)}\kappa_2(\eta_3, t)\,(s-t)}{s - \tau_j^0} + \frac{D^{(1,0)}\kappa_1(\eta_4, t)\,(t - \tau_j^0)}{s - \tau_j^0},\end{aligned}$$

for some $\eta_3 \in (s, t)$ and $\eta_4 \in (t, \tau_j^0)$. Taking absolute values and using the boundedness of derivatives, we have

$$\left|\frac{\kappa_2(s,t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0}\right| \le \left|D^{(1,0)}\kappa_2(\eta_3, t)\right| + \left|D^{(1,0)}\kappa_1(\eta_4, t)\right| \le 2M_1.$$

Therefore, we write

$$\left|[\tau_j^0, s]\kappa(\cdot, t)\right| \le 2M_1, \quad \text{for } s \le t \le \tau_j^0.$$

Thus, combining this with (2.3.4), we conclude that

$$\sup_{t \in [t_{j-1}, t_j]} \left|[\tau_j^0, s]\kappa(\cdot, t)\right| \le 2M_1, \quad \text{for all } s \in [t_{j-1}, \tau_j^0]. \tag{2.3.5}$$

Case (II): $s = \tau_j^0$. In this case, by definition,

$$[\tau_j^0, \tau_j^0]\kappa(\cdot, t) = \begin{cases} D^{(1,0)}\kappa_1(\tau_j^0, t), & t_{j-1} \le t < \tau_j^0, \\ D^{(1,0)}\kappa_2(\tau_j^0, t), & \tau_j^0 < t \le t_j, \end{cases}$$

and hence

$$\sup_{t\in[t_{j-1},t_j]} \left|[\tau_j^0,\tau_j^0]\kappa(\cdot,t)\right| \le M_1. \tag{2.3.6}$$

Case (III): $\tau_j^0 < s \le t_j$. Proceeding analogously as in Case (I), we again obtain

$$\sup_{t\in[t_{j-1},t_j]} \left|[\tau_j^0,s]\kappa(\cdot,t)\right| \le 2M_1.$$

Combining (2.3.5), (2.3.6), and above estimate, we conclude that

$$\sup_{s,t\in[t_{j-1},t_j]} \left|[\tau_j^0,s]\kappa(\cdot,t)\right| \le 2M_1. \tag{2.3.7}$$

Now, for the second divided difference, using the definition we have

$$[\tau_j^0,\tau_j^1,s]\kappa(\cdot,t) = \frac{[\tau_j^0,s]\kappa(\cdot,t)-[\tau_j^1,s]\kappa(\cdot,t)}{\tau_j^0-\tau_j^1} = \frac{1}{h(\zeta_0-\zeta_1)}\Big([\tau_j^0,s]\kappa(\cdot,t)-[\tau_j^1,s]\kappa(\cdot,t)\Big).$$

Hence, it follows from (2.3.7) that

$$\left|[\tau_j^0,\tau_j^1,s]\kappa(\cdot,t)\right| \le \frac{1}{h|\zeta_0-\zeta_1|}\Big(\left|[\tau_j^0,s]\kappa(\cdot,t)\right|+\left|[\tau_j^1,s]\kappa(\cdot,t)\right|\Big) \le \frac{4M_1}{h|\zeta_0-\zeta_1|}.$$

Therefore,

$$\left|\int_{t_{j-1}}^{t_j}[\tau_j^0,\tau_j^1,s]\kappa(\cdot,t)\,x(t)\,dt\right| \le \frac{4M_1}{h|\zeta_0-\zeta_1|}\,\|x\|_\infty = \frac{C_8}{h}\,\|x\|_\infty,$$

where $C_8 = \dfrac{4M_1}{|\zeta_0-\zeta_1|}$ is a constant independent of h.

Finally, combining (2.3.3) with the above inequality in (2.3.2), we get

$$\left|[\tau_j^0,\tau_j^1,s]\mathcal{K}x\right| \le \left(M_2 h + \frac{C_8}{h}\right)\|x\|_\infty \le \frac{C_9}{h}\|x\|_\infty,$$

for some constant $C_9 > 0$. By extending this reasoning inductively to higher-order divided differences, we obtain the desired estimate

$$\left|[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},s]\mathcal{K}x\right| \le C_{10}\|x\|_\infty h^{-2r},$$

for some constant $C_{10} > 0$ independent of h. Since $s \in [t_{j-1},t_j]$ was arbitrary, the proof is complete. □

Lemma 2.3.2. *Let $r \ge 0$ and let $\left\{\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r}\right\}$ be the set of collocation points in the subinterval $[t_{j-1},t_j]$ for $j = 1,2,\ldots,n$. Then, for some constant C_{13}, we have*

$$\sup_{s\in[t_{j-1},t_j]} \left|\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},s,s\right]\mathcal{K}x\right| \le C_{13}\,\|x\|_\infty\,h^{-2r}.$$

Proof. Let $s \in [t_{j-1}, t_j]$. Using the linearity of the integral operator $\mathcal{K}$, we can write

$$\begin{aligned}[\tau_j^0, s, s]\mathcal{K}x &= \int_0^1 [\tau_j^0, s, s]\kappa(\cdot, t)\, x(t)\, dt \\ &= \sum_{\substack{k=1\\k\neq j}}^{n} \int_{t_{k-1}}^{t_k} [\tau_j^0, s, s]\kappa(\cdot, t)\, x(t)\, dt + \int_{t_{j-1}}^{t_j} [\tau_j^0, s, s]\kappa(\cdot, t)\, x(t)\, dt. \end{aligned} \tag{2.3.8}$$

Since the kernel κ is assumed to be sufficiently differentiable with respect to s in each subinterval $[t_{k-1}, t_k]$ for $k \neq j$, the first term in (2.3.8) can be bounded as

$$\left| \int_{t_{k-1}}^{t_k} [\tau_j^0, s, s]\kappa(\cdot, t)\, x(t)\, dt \right| \leq M_2 \|x\|_\infty h. \tag{2.3.9}$$

The main effort now is to estimate the second integral in (2.3.8), corresponding to the local interval $[t_{j-1}, t_j]$. To this end, we analyze $[\tau_j^0, s, s]\kappa(\cdot, t)$ for $\tau_j^0, s \in [t_{j-1}, t_j]$.

Case (I): $t_{j-1} \leq s < \tau_j^0$. In this case, by the definition of divided differences,

$$[\tau_j^0, s, s]\kappa(\cdot, t) = \frac{[s, s]\kappa(\cdot, t) - [\tau_j^0, s]\kappa(\cdot, t)}{s - \tau_j^0}.$$

Hence, expanding both terms according to the piecewise nature of κ, we obtain

$$[\tau_j^0, s, s]\kappa(\cdot, t) = \begin{cases} \dfrac{\dfrac{\partial \kappa_1}{\partial s}(s, t) - \left(\dfrac{\kappa_1(s, t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} \right)}{s - \tau_j^0}, & t_{j-1} \leq t \leq s, \\ \dfrac{\dfrac{\partial \kappa_2}{\partial s}(s, t) - \left(\dfrac{\kappa_2(s, t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} \right)}{s - \tau_j^0}, & s < t < \tau_j^0, \\ \dfrac{\dfrac{\partial \kappa_2}{\partial s}(s, t) - \left(\dfrac{\kappa_2(s, t) - \kappa_2(\tau_j^0, t)}{s - \tau_j^0} \right)}{s - \tau_j^0}, & \tau_j^0 \leq t \leq t_j. \end{cases}$$

The function $[\tau_j^0, s, s]\kappa(\cdot, t)$ may have a possible discontinuity at $t = s$ due to the piecewise definition of κ. However, since both $\kappa_1(\cdot, t)$ and $\kappa_2(\cdot, t)$ are continuous on $[s, \tau_j^0]$ and differentiable on (s, τ_j^0), the Mean Value Theorem implies that

$$\frac{\kappa_1(s, t) - \kappa_1(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\kappa_1(\nu_1, t), \qquad \frac{\kappa_2(s, t) - \kappa_2(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\kappa_2(\nu_2, t),$$

for some $\nu_1, \nu_2 \in (s, \tau_j^0)$. Then, for $t_{j-1} \leq t \leq s$, we have

$$\left|\frac{\frac{\partial \kappa_1}{\partial s}(s,t) - \left(\frac{\kappa_1(s,t)-\kappa_1(\tau_j^0,t)}{s-\tau_j^0}\right)}{s-\tau_j^0}\right| = \left|\frac{D^{(1,0)}\kappa_1(s,t) - D^{(1,0)}\kappa_1(\nu_1,t)}{s-\tau_j^0}\right|$$
$$= \left|\frac{D^{(2,0)}\kappa_1(\nu_3,t)(s-\nu_1)}{s-\tau_j^0}\right|, \quad \text{for some } \nu_3 \in (s, \nu_1).$$

Since $D^{(2,0)}\kappa_1$ is continuous and bounded, we write

$$\left|[\tau_j^0, s, s]\kappa(\cdot, t)\right| \leq M_2, \quad \text{for all } t_{j-1} \leq t \leq s.$$

Similarly, for $\tau_j^0 \leq t \leq t_j$, an analogous argument yields

$$\left|[\tau_j^0, s, s]\kappa(\cdot, t)\right| \leq M_2.$$

Thus,

$$\left|[\tau_j^0, s, s]\kappa(\cdot, t)\right| \leq M_2, \quad \text{for } t_{j-1} \leq t \leq s \text{ and } \tau_j^0 \leq t \leq t_j. \tag{2.3.10}$$

Next, for the intermediate range $s < t < \tau_j^0$, we write

$$\frac{\kappa_2(s,t) - \kappa_1(\tau_j^0,t)}{s-\tau_j^0} = \frac{\kappa_2(s,t) - \kappa_2(t,t) + \kappa_1(t,t) - \kappa_1(\tau_j^0,t)}{s-\tau_j^0}$$
$$= \frac{D^{(1,0)}\kappa_2(\nu_5,t)(s-t)}{s-\tau_j^0} + \frac{D^{(1,0)}\kappa_1(\nu_6,t)(t-\tau_j^0)}{s-\tau_j^0},$$

for some $\nu_5 \in (s,t)$ and $\nu_6 \in (t, \tau_j^0)$. Then

$$\left|\frac{\partial \kappa_2}{\partial s}(s,t) - \left(\frac{\kappa_2(s,t)-\kappa_1(\tau_j^0,t)}{s-\tau_j^0}\right)\right|$$
$$= \left|D^{(1,0)}\kappa_2(s,t) - \frac{D^{(1,0)}\kappa_2(\nu_5,t)(s-t)}{s-\tau_j^0} - \frac{D^{(1,0)}\kappa_1(\nu_6,t)(t-\tau_j^0)}{s-\tau_j^0}\right|$$

gives

$$\left|\frac{\partial \kappa_2}{\partial s}(s,t) - \left(\frac{\kappa_2(s,t)-\kappa_1(\tau_j^0,t)}{s-\tau_j^0}\right)\right| \leq M_1 + M_1 + M_1 = 3M_1.$$

Therefore,

$$\int_s^{\tau_j^0} \left|\frac{\frac{\partial \kappa_2}{\partial s}(s,t) - \left(\frac{\kappa_2(s,t)-\kappa_1(\tau_j^0,t)}{s-\tau_j^0}\right)}{s-\tau_j^0}\right| dt \leq 3M_1.$$

Combining this with (2.3.10), we have

$$\left|\int_{t_{j-1}}^{t_j} [\tau_j^0, s, s]\kappa(\cdot,t)x(t)\,dt\right| \leq (M_2 + 3M_1 + M_2)\,\|x\|_\infty = (3M_1 + 2M_2)\|x\|_\infty.$$

Finally, substituting this and (2.3.9) into (2.3.8), we obtain

$$\left|[\tau_j^0, s, s]\mathcal{K}x\right| \leq (M_2 h + 3M_1 + 2M_2)\, \|x\|_\infty \leq C_{11}\|x\|_\infty, \tag{2.3.11}$$

where $C_{11} = 3M_1 + 3M_2$ is a constant independent of h, obtained by noting that $M_2 h \leq M_2$ for $0 < h \leq 1$ and hence absorbing the h-dependent term into the constant bound.

Case (II): $s = \tau_j^0$. Here,

$$\begin{aligned}
[\tau_j^0, \tau_j^0, \tau_j^0]\mathcal{K}x &= \frac{1}{2}(\mathcal{K}x)''(\tau_j^0) \\
&= \frac{1}{2}\left(\frac{\partial \kappa_1}{\partial s}(\tau_j^0, \tau_j^0) - \frac{\partial \kappa_2}{\partial s}(\tau_j^0, \tau_j^0)\right) x(\tau_j^0) \\
&\quad + \frac{1}{2}\left[\int_{t_{j-1}}^{\tau_j^0} \frac{\partial^2 \kappa_1}{\partial s^2}(\tau_j^0, t)x(t)\, dt + \int_{\tau_j^0}^{t_j} \frac{\partial^2 \kappa_2}{\partial s^2}(\tau_j^0, t)x(t)\, dt\right].
\end{aligned}$$

Hence,

$$\left|[\tau_j^0, \tau_j^0, \tau_j^0]\mathcal{K}x\right| \leq (M_1 + M_2)\|x\|_\infty. \tag{2.3.12}$$

Case (III): $\tau_j^0 < s \leq t_j$. By symmetry, an argument identical to Case (I) yields

$$\left|[\tau_j^0, s, s]\mathcal{K}x\right| \leq (M_2 h + 3M_1 + 2M_2)\|x\|_\infty \leq C_{11}\|x\|_\infty.$$

Combining (2.3.11), (2.3.12), and the above bound, we obtain

$$\sup_{s\in[t_{j-1}, t_j]} \left|[\tau_j^0, s, s]\mathcal{K}x\right| \leq C_{11}\|x\|_\infty, \tag{2.3.13}$$

which proves the desired estimate for $r = 0$.

For $r = 1$, we have

$$[\tau_j^0, \tau_j^1, s, s]\mathcal{K}x = \frac{[\tau_j^0, s, s]\mathcal{K}x - [\tau_j^1, s, s]\mathcal{K}x}{\tau_j^0 - \tau_j^1} = \frac{1}{h(\zeta_0 - \zeta_1)}\Big([\tau_j^0, s, s]\mathcal{K}x - [\tau_j^1, s, s]\mathcal{K}x\Big),$$

which gives

$$\left|[\tau_j^0, \tau_j^1, s, s]\mathcal{K}x\right| \leq \frac{1}{h|\zeta_0 - \zeta_1|}\Big(\left|[\tau_j^0, s, s]\mathcal{K}x\right| + \left|[\tau_j^1, s, s]\mathcal{K}x\right|\Big).$$

Applying (2.3.13), it follows that

$$\sup_{s\in[t_{j-1}, t_j]} \left|[\tau_j^0, \tau_j^1, s, s]\mathcal{K}x\right| \leq \frac{2C_{11}}{h|\zeta_0 - \zeta_1|}\|x\|_\infty = C_{12}h^{-1}\|x\|_\infty,$$

where $C_{12} = \dfrac{2C_{11}}{|\zeta_0 - \zeta_1|}$.

Finally, applying this argument recursively for higher-order divided differences, by mathematical induction we conclude that

$$\sup_{s\in[t_{j-1},t_j]} \left|[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s, s]\mathcal{K}x\right| \leq C_{13}\|x\|_\infty h^{-2r},$$

where C_{13} is a positive constant independent of h. This completes the proof. □

2.3.2 Error Estimates for Green's Kernels

We now proceed to establish several auxiliary results using standard analytical techniques together with the previously stated lemmas. These results will play a crucial role in deriving the main theorem.

Proposition 18. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq 2C_7\|x\|_{2r+2,\infty}h^{2r+2},$$

where C_7 is a constant, defined earlier, and independent of h.

Proof. For any fixed $s \in [0,1]$, the action of $\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)$ can be written as

$$\begin{aligned}
\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x(s) &= \int_0^1 \kappa(s,t)(\mathcal{I} - \mathcal{Q}_n)x(t)\,dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \kappa(s,t)(I - \mathcal{Q}_{n,j})x(t)\,dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \kappa(s,t)\Psi_j(t)[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t]x\,dt,
\end{aligned}$$

where

$$\Psi_j(t) = \prod_{i=0}^{2r}(t - \tau_j^i), \quad j = 1, 2, \ldots, n.$$

Now, fix $s \in [t_{i-1}, t_i] \subset [0,1]$ for some $1 \leq i \leq n$. We decompose the summation in two parts to distinguish the integral over the element containing s:

$$\begin{aligned}
\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x(s) = &\sum_{\substack{j=1\\ j\neq i}}^{n} \int_{t_{j-1}}^{t_j} \kappa(s,t)\Psi_j(t)[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t]x\,dt \\
&+ \int_{t_{i-1}}^{t_i} \kappa(s,t)\Psi_i(t)[\tau_i^0, \tau_i^1, \ldots, \tau_i^{2r}, t]x\,dt. \qquad (2.3.14)
\end{aligned}$$

For each fixed s, define the function

$$f_j^s(t) = \kappa(s,t)[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t]x, \quad t \in [t_{j-1}, t_j],$$

so that each term in (2.3.14) can be expressed as an integral involving $f_j^s(t)\Psi_j(t)$. Depending on the position of the subinterval relative to s, we write

$$f_j^s(t) = \begin{cases} \kappa_1(s,t)[\tau_j^0, \ldots, \tau_j^{2r}, t]x, & j \le i, \\ \kappa_2(s,t)[\tau_j^0, \ldots, \tau_j^{2r}, t]x, & j \ge i. \end{cases}$$

Since κ is continuously differentiable in t and $x \in \mathcal{C}^{2r+2}[0,1]$, each f_j^s is continuous on $[t_{j-1}, t_j]$ and differentiable on (t_{j-1}, t_j).

Applying the Taylor expansion of $f_j^s(t)$ around the midpoint c_j of Δ_j, there exists $\zeta_{j1} \in (t_{j-1}, t_j)$ such that

$$f_j^s(t) = f_j^s(c_j) + (t - c_j)\,(f_j^s)'(\zeta_{j1}).$$

Differentiating $f_j^s(t)$ yields

$$(f_j^s)'(\zeta_{j1}) = \begin{cases} \kappa_1(s,\zeta_{j1})[\tau_j^0, \ldots, \tau_j^{2r}, \zeta_{j1}, \zeta_{j1}]x + D^{(0,1)}\kappa_1(s,\zeta_{j1})[\tau_j^0, \ldots, \tau_j^{2r}, \zeta_{j1}]x, & j \le i, \\ \kappa_2(s,\zeta_{j1})[\tau_j^0, \ldots, \tau_j^{2r}, \zeta_{j1}, \zeta_{j1}]x + D^{(0,1)}\kappa_2(s,\zeta_{j1})[\tau_j^0, \ldots, \tau_j^{2r}, \zeta_{j1}]x, & j \ge i. \end{cases}$$

Since κ and its partial derivatives up to first order are bounded on $[0,1] \times [0,1]$, and since the divided differences of x up to order $2r+2$ are bounded by $\|x\|_{2r+2,\infty}$, we have

$$\left|(f_j^s)'(\zeta_{j1})\right| \le 2C_7\|x\|_{2r+2,\infty}.$$

Hence,

$$\int_{t_{j-1}}^{t_j} \kappa(s,t)\Psi_j(t)[\tau_j^0, \ldots, \tau_j^{2r}, t]x\,dt = \int_{t_{j-1}}^{t_j} (t - c_j)\,(f_j^s)'(\zeta_{j1})\Psi_j(t)\,dt.$$

The integral on the right-hand side is of order h^{2r+3} because $\Psi_j(t)$ is a polynomial of degree $2r+1$ and $(t - c_j)$ adds one more power of h. Therefore,

$$\left|\int_{t_{j-1}}^{t_j} \kappa(s,t)\Psi_j(t)[\tau_j^0, \ldots, \tau_j^{2r}, t]x\,dt\right| \le 2C_7\|x\|_{2r+2,\infty}h^{2r+3}.$$

For the interval containing s, namely $[t_{i-1}, t_i]$, by Lemma 1.3.1, we have

$$\int_{t_{i-1}}^{t_i} \Psi_i(t)\,dt = 0.$$

Hence, we can write

$$\begin{aligned}\int_{t_{i-1}}^{t_i} \kappa(s,t)\Psi_i(t)\left[\tau_i^0, \tau_i^1, \ldots, \tau_i^{2r}, t\right] x \; dt &= \int_{t_{i-1}}^{t_i} (f_i^s)(t)\Psi_i(t) \; dt, \quad s \in [t_{i-1}, t_i] \\ &= \int_{t_{i-1}}^{t_i} [(f_i^s)(t) - (f_i^s)(s)]\Psi_i(t) \; dt.\end{aligned}$$

Since the function f_i^s is continuous on $[t_{i-1}, t_i]$ and differentiable on (t_{i-1}, t_i), by Mean Value Theorem, for some $\zeta_{j2} \in (t_{i-1}, t_i)$, we have

$$(f_i^s)(t) - (f_i^s)(s) = (t-s)(f_i^s)'(\zeta_{j2}).$$

Consequently,

$$\left| \int_{t_{i-1}}^{t_i} \kappa(s,t)\Psi_i(t)[\tau_i^0, \ldots, \tau_i^{2r}, t]x \, dt \right| \leq 2C_7 \|x\|_{2r+2,\infty} h^{2r+3}.$$

Combining both cases in (2.3.14), we obtain

$$\begin{aligned}|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x(s)| \leq & \sum_{\substack{j=1 \\ j \neq i}}^{n} \left| \int_{t_{j-1}}^{t_j} \kappa(s,t)[\tau_j^0, \ldots, \tau_j^{2r}, t]x \; \Psi_j(t) \, dt \right| \\ & + \left| \int_{t_{i-1}}^{t_i} \kappa(s,t)[\tau_i^0, \ldots, \tau_i^{2r}, t]x \; \Psi_i(t) \, dt \right|.\end{aligned}$$

Since there are n such subintervals with uniform width $h = \dfrac{1}{n}$, the summation over all j yields a total contribution proportional to h^{2r+2}. Consequently, we obtain

$$|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x(s)| \leq \sum_{j=1}^{n} 2C_7 \|x\|_{2r+2,\infty} h^{2r+3} \leq 2C_7 \|x\|_{2r+2,\infty} h^{2r+2}.$$

Taking the supremum over $s \in [0,1]$, the desired inequality follows:

$$\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq 2C_7 \|x\|_{2r+2,\infty} h^{2r+2},$$

which completes the proof. □

Proposition 19. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x\|_\infty = \begin{cases} O(h^{2r+2}), & r \geq 1, \\ O(h^3), & r = 0. \end{cases}$$

Proof. We begin by considering the case $r \geq 1$. For any sufficiently smooth function $x \in \mathcal{C}^{2r+2}[0,1]$, we can estimate the operator norm as

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq \|\mathcal{I}-\mathcal{Q}_n\| \, \|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty.$$

Since the sequence of projection operators $\{\mathcal{Q}_n\}$ is uniformly bounded, there exists a positive constant $q > 0$ such that $\|\mathcal{Q}_n\| \leq q$ for all n. Consequently, $\|\mathcal{I}-\mathcal{Q}_n\| \leq 1+q$, which is also bounded independently of n. Now, by Proposition 18, we already know that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq 2C_7\|x\|_{2r+2,\infty}h^{2r+2}.$$

Substituting this estimate into the previous inequality yields

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq 2C_7(1+q)\|x\|_{2r+2,\infty}h^{2r+2}.$$

Hence,

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\big(h^{2r+2}\big)\,, \quad \text{for } r \geq 1.$$

Next, we treat the case $r = 0$. In this situation, $\mathcal{Q}_n$ corresponds to a piecewise constant projection operator. Using inequality (2.1.8), we can write

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq C_1 \, \|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)'\|_\infty \, h.$$

From the result of Chatelin-Lebbar [22, Theorem 15], it follows that the derivative of $\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x$ satisfies the bound

$$\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x)'\|_\infty = O\big(h^2\big)\,.$$

Combining the above two estimates, we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\big(h^3\big)\,, \quad \text{for } r = 0.$$

This concludes the proof. □

Proposition 20. *If $x \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = \begin{cases} O(h^{2r+3})\,, & r \geq 1, \\ O(h^4)\,, & r = 0. \end{cases}$$

Proof. We begin by observing that for any $s \in [0,1]$, the operator expression can be written as

$$
\begin{aligned}
\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) &= \int_0^1 \kappa(s,t)\,(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \kappa(s,t)\,(I-\mathcal{Q}_{n,j})\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt \\
&= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \kappa(s,t)\,\left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)\Psi_j(t)\,dt.
\end{aligned}
\tag{2.3.15}
$$

Let $s \in [t_{i-1}, t_i] \subset [0,1]$ for some fixed i with $1 \le i \le n$. For convenience, define for each subinterval $[t_{j-1}, t_j]$

$$
g_j^s(t) = \kappa(s,t)\,\left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right), \quad t \in [t_{j-1}, t_j],
$$

for $j = 1, 2, \ldots, n$. Depending on whether $t \le s$ or $t \ge s$, the kernel splits naturally as

$$
g_j^s(t) = \begin{cases} \kappa_1(s,t)\,\left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right), & j \le i, \\ \kappa_2(s,t)\,\left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right), & j \ge i. \end{cases}
$$

We denote the divided differences by

$$
\begin{aligned}
\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t] &= \left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right), \\
\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] &= \left([\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t,t]\,\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right).
\end{aligned}
$$

For $j \ne i$, the function g_j^s is continuous on $[t_{j-1}, t_j]$ and differentiable on (t_{j-1}, t_j). Hence, differentiating $g_j^s(t)$ yields

$$
(g_j^s)'(t) = \begin{cases} \kappa_1(s,t)\,\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] + D^{(0,1)}\kappa_1(s,t)\,\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t], & j \le i, \\ \kappa_2(s,t)\,\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] + D^{(0,1)}\kappa_2(s,t)\,\left(\delta_j^{2r}\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\right)[t], & j \ge i. \end{cases}
$$

By applying Lemma 2.3.1 and Lemma 2.3.2, together with the interpolation estimate (2.1.8), we obtain a uniform bound on the derivative of $g_j^s(t)$:

$$
\left\|(g_j^s)'(t)\right\|_\infty \le C_{14}\,C_7\,\|(\mathcal{I}-\mathcal{Q}_n)x\|_\infty\,h^{-2r} \le C_{14}\,C_7\,C_1\,\left\|x^{(2r+1)}\right\|_\infty h,
$$

where C_{14} is a positive constant independent of h.

Substituting $g_j^s(t)$ back into (2.3.15), we can rewrite

$$\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_{t_{j-1}}^{t_j} g_j^s(t)\,\Psi_j(t)\,dt + \int_{t_{i-1}}^{t_i} g_i^s(t)\,\Psi_i(t)\,dt.$$

Following a similar argument as used in the proof of Proposition 18, and using the above derivative bound, we derive that

$$|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x(s)| \leq 2\,C_{14}\,C_7\,C_1 \left\|x^{(2r+1)}\right\|_\infty h^{2r+3}.$$

Hence, for $r \geq 1$,

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\left(h^{2r+3}\right).$$

For the case $r = 0$ and $x \in \mathcal{C}^2[0,1]$, we use the result from Kulkarni-Nidhin [57, Proposition 3.7], which establishes that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty = O\left(h^4\right).$$

This completes the proof of the proposition. □

2.3.3 Collocation Methods and Their Variants

In this subsection, we establish the convergence behaviour of various collocation-type methods when the kernel of the given Fredholm integral equation is of Green's function-type. The analysis includes both the standard and modified collocation methods along with their corresponding iterated versions.

We begin by determining the orders of convergence for the collocation solution φ_n^C and its iterated form φ_n^S. The results are based on classical error estimates available in the literature, particularly in Atkinson [13, Section 3.4]; see Chapter 1.

Theorem 21. *Let $\mathcal{K}$ be the Fredholm integral operator defined by equation* (2.1.1) *with a Green's function-type kernel, and let φ denote the unique solution of the corresponding integral equation* (2.1.2) *for $f \in \mathcal{C}^{2r+2}[0,1]$. Let $\mathcal{Q}_n$ denote the interpolatory projection operator onto the finite-dimensional approximating space $\mathcal{X}_n$ defined in* (2.1.7)*. If φ_n^C and φ_n^S are the collocation and iterated collocation approximations of φ, respectively, defined by* (2.1.3) *and* (2.1.4)*, then we have*

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\left(h^{2r+1}\right), \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\left(h^{2r+2}\right).$$

Proof. The estimate (2.2.5) indicates that the collocation error is bounded by the interpolation error of the exact solution φ. Using the approximation property of the interpolatory operator $\mathcal{Q}_n$ and inequality (2.1.8), we know that

$$\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = O\big(h^{2r+1}\big)\,.$$

Using this in (2.2.5) immediately gives

$$\big\|\varphi - \varphi_n^C\big\|_\infty = O\big(h^{2r+1}\big)\,.$$

Similarly, by applying Proposition 18, which provides an additional order of improvement due to the smoothing property of the integral operator $\mathcal{K}$ in (2.2.6), we obtain

$$\big\|\varphi - \varphi_n^S\big\|_\infty = O\big(h^{2r+2}\big)\,.$$

Hence, both results follow, completing the proof. $\square$

The above theorem clearly shows that the collocation approximation φ_n^C achieves an order of convergence $O(h^{2r+1})$, while its iterated counterpart φ_n^S achieves a higher order $O(h^{2r+2})$. This theorem therefore provides a rigorous theoretical justification for employing the iterated collocation approach to obtain faster convergence without increasing the degree of the approximating polynomial space.

We now turn to the modified collocation method, which is designed to improve the accuracy of the standard collocation solution. The convergence properties of both the modified collocation solution φ_n^M and its iterated form $\widetilde{\varphi}_n^M$ are established below.

Theorem 22. *Let $\mathcal{K}$ be the Fredholm integral operator defined by* (2.1.1) *with a Green's function-type kernel. For $f \in \mathcal{C}^{2r+2}[0,1]$, let φ denote the unique solution of* (2.1.2)*, and let $\mathcal{Q}_n$ be the interpolatory projection onto $\mathcal{X}_n$ defined in* (2.1.7)*. Let φ_n^M and $\widetilde{\varphi}_n^M$ denote the modified collocation and iterated modified collocation solutions defined by* (2.1.5) *and* (2.1.6)*, respectively. Then we have*

$$\big\|\varphi - \varphi_n^M\big\|_\infty = \begin{cases} O(h^{2r+2})\,, & r \geq 1, \\ O(h^3)\,, & r = 0, \end{cases}$$

and

$$\big\|\varphi - \widetilde{\varphi}_n^M\big\|_\infty = \begin{cases} O(h^{2r+3})\,, & r \geq 1, \\ O(h^4)\,, & r = 0. \end{cases}$$

Proof. From (2.2.7), the error in the modified collocation approximation satisfies

$$\left\|\varphi - \varphi_n^M\right\|_\infty \leq C_6 \left\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\right\|_\infty.$$

By applying Proposition 19, which provides an estimate for the double application of the operator $\mathcal{K}$ with the projection error, we have

$$\left\|\varphi - \varphi_n^M\right\|_\infty = \begin{cases} O(h^{2r+2}), & r \geq 1, \\ O(h^3), & r = 0. \end{cases}$$

Next, we consider the iterated modified collocation solution $\widetilde{\varphi}_n^M$. From (2.2.8), the corresponding error satisfies the inequality

$$\begin{aligned} \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty \leq& \left\|(\mathcal{I} - \mathcal{K})^{-1}\right\| \left\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\varphi\right\|_\infty \\ &+ \left\|(\mathcal{I} - \mathcal{K})^{-1}\right\| \left\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\right\| \left\|\varphi - \varphi_n^M\right\|_\infty. \end{aligned} \tag{2.3.16}$$

Since $\mathcal{K}$ is a bounded linear operator, and using inequality (2.1.8), we obtain the following bound:

$$\left\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x\right\|_\infty \leq C_1 \|\mathcal{K}\| \left\|(\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x)'\right\|_\infty h.$$

Using inequality (2.3.1) and boundedness of the operator $\mathcal{Q}_n$, we obtain

$$\left\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)x\right\|_\infty \leq C_7 C_1 (1+q) \|\mathcal{K}\| \|x\|_\infty h,$$

which shows that

$$\left\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\right\| = O(h).$$

Substituting this estimate and the earlier result for $\left\|\varphi - \varphi_n^M\right\|_\infty$ into inequality (2.3.16) and using Proposition 20, we conclude that

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = \begin{cases} O(h^{2r+3}), & r \geq 1, \\ O(h^4), & r = 0. \end{cases}$$

Hence, the theorem is proved. □

The above theorem shows that the modified and iterated modified collocation methods achieve higher-order convergence, with the iterated version providing further improvement. This reflects their superconvergent behaviour and effectiveness in accurately approximating solutions of Fredholm integral equations with non-smooth Green's kernels.

In this section, we have analyzed the collocation method and its variants for approximating the solution of the Fredholm integral equation (2.1.2) with a Green's function-type kernel. The sequence of projections was defined through interpolatory operators at $2r+1$ non-Gaussian collocation points, based on a uniform partition of $[0,1]$. Theorems 21 and 22 establish that, under this framework, both the iterated collocation and iterated modified collocation methods exhibit enhanced convergence orders. These results confirm that the proposed projection approach ensures accelerated convergence even without relying on Gauss points, thereby providing an effective and flexible alternative for obtaining accurate numerical solutions.

2.4 Numerical Results

We now present numerical experiments to verify the theoretical convergence rates of the collocation, iterated collocation, modified collocation, and iterated modified collocation methods developed in the previous sections.

Let $\mathcal{X}_n$ denote the space of piecewise constant functions defined with respect to the uniform partition of the interval $[0,1]$:

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n.$$

The interpolatory projection $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ is defined by

$$\mathcal{Q}_n x\left(\frac{2j-1}{2n}\right) = x\left(\frac{2j-1}{2n}\right), \quad j = 1, 2, \ldots, n.$$

We observe that, for the iterated collocation method, the order of convergence is $\frac{1}{n^2}$, whereas for the iterated version of the modified collocation method, it improves to $\frac{1}{n^4}$. Consequently, to evaluate certain integrals numerically with sufficient accuracy, a quadrature rule possessing at least $\frac{1}{n^4}$ convergence is required.

We define the error quantities corresponding to the numerical solutions obtained by solving linear systems of sizes n and $2n$, respectively, as

$$e_n = \|\varphi - \varphi_n\|_\infty, \quad e_{2n} = \|\varphi - \varphi_{2n}\|_\infty.$$

Using these error measures, the observed order of convergence is computed as

$$\delta = \frac{\log\left(e_n/e_{2n}\right)}{\log 2}.$$

In the following tables, the quantities δ_C, δ_{IC}, δ_M, and δ_{IM} represent the computed orders of convergence corresponding to the collocation solution φ_n^C, the iterated collocation solution φ_n^S, the modified collocation solution φ_n^M, and the iterated modified collocation solution $\widetilde{\varphi}_n^M$, respectively. According to the theoretical results, the expected orders of convergence are

$$\delta_C = 1, \quad \delta_{IC} = 2, \quad \delta_M = 3, \quad \delta_{IM} = 4.$$

Smooth kernel case: We first present two examples of Fredholm integral equations with smooth kernels to illustrate the theoretical results developed above.

Example 2. *Consider the Fredholm integral equation of the second kind,*

$$x(s) - \int_0^1 \kappa(s,t)\, x(t)\, dt = f(s), \quad s \in [0,1].$$

The kernel $\kappa(s,t) = \dfrac{1}{2}e^{st}$ *is infinitely smooth (analytic) on* Ω. *The right-hand side function* $f(s)$ *is chosen such that the exact solution is given in closed form by* $\varphi(s) = e^{-s}\cos(s)$.

Table 2.4.1: Comparison of convergence rates in collocation variants for Example 2.

n	$\lVert\varphi - \varphi_n^C\rVert_\infty$	δ_C	$\lVert\varphi - \varphi_n^S\rVert_\infty$	δ_{IC}	$\lVert\varphi - \varphi_n^M\rVert_\infty$	δ_M	$\lVert\varphi - \widetilde{\varphi}_n^M\rVert_\infty$	δ_{IM}
2	2.42×10^{-1}		1.56×10^{-2}		4.66×10^{-3}		7.90×10^{-4}	
4	1.24×10^{-1}	0.97	4.08×10^{-3}	1.94	5.66×10^{-4}	3.04	5.04×10^{-5}	3.97
8	6.23×10^{-2}	0.99	1.03×10^{-3}	1.98	6.96×10^{-5}	3.02	3.16×10^{-6}	3.99
16	3.12×10^{-2}	1.00	2.59×10^{-4}	2.00	8.63×10^{-6}	3.01	1.98×10^{-7}	4.00
32	1.56×10^{-2}	1.00	6.47×10^{-5}	2.00	1.07×10^{-6}	3.01	1.24×10^{-8}	4.00
64	7.81×10^{-3}	1.00	1.62×10^{-5}	2.00	1.34×10^{-7}	3.00	7.74×10^{-10}	4.00

The results in Table 2.4.1 clearly demonstrate the progressive improvement in convergence obtained through iteration and modified projection methods. The standard collocation method exhibits first-order convergence, while the iterated collocation method improves the rate to second order. The modified collocation method achieves third-order convergence, and the iterated modified method attains the highest fourth-order accuracy.

Example 3. *Consider the Fredholm integral equation of the second kind,*

$$x(s) - (\mathcal{K}x)(s) = 1 + s^{5/2}, \quad s \in [0,1],$$

where the compact integral operator $\mathcal{K}$ is defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt,$$

and $\kappa(s,t) = \dfrac{1}{1+(s-t)^2}$ *is a smooth kernel on* $[0,1]\times[0,1]$. *The exact solution* $\varphi(s)$ *is not known in closed form; however, it is known that* $\varphi \in \mathcal{C}^2[0,1]$ *but* $\varphi \notin \mathcal{C}^3[0,1]$.

This example was originally investigated by Kulkarni [48] and the results reported therein, as well as the numerical data presented in Table 2.4.2, confirm the theoretically predicted orders of convergence. In particular, the collocation, iterated collocation, modified collocation, and iterated modified collocation methods exhibit convergence of orders 1, 2, 3, and 4, respectively.

Table 2.4.2: Comparison of convergence rates in collocation variants for Example 3.

n	$\lVert\varphi - \varphi_n^C\rVert_\infty$	δ_C	$\lVert\varphi - \varphi_n^S\rVert_\infty$	δ_{IC}	$\lVert\varphi - \varphi_n^M\rVert_\infty$	δ_M	$\lVert\varphi - \widetilde{\varphi}_n^M\rVert_\infty$	δ_{IM}
4	7.57×10^{-1}		1.31×10^{-1}		1.26×10^{-3}		1.66×10^{-5}	
8	3.57×10^{-1}	1.08	3.19×10^{-2}	2.04	1.59×10^{-4}	2.99	1.29×10^{-6}	3.68
16	1.74×10^{-1}	1.04	7.93×10^{-3}	2.01	1.98×10^{-5}	3.00	8.54×10^{-8}	3.92
32	8.56×10^{-2}	1.02	1.98×10^{-3}	2.00	2.49×10^{-6}	3.00	5.48×10^{-9}	3.96
64	4.25×10^{-2}	1.01	4.95×10^{-4}	2.00	3.11×10^{-7}	3.00	3.50×10^{-10}	3.97
128	2.12×10^{-2}	1.01	1.24×10^{-4}	2.00	3.89×10^{-8}	3.00	2.12×10^{-11}	4.04

Green's kernel case: We next present two illustrative examples involving Fredholm integral equations with Green's function-type kernels in order to further examine the performance of the proposed methods.

Example 4. *Consider the Fredholm integral equation of the second kind,*

$$x(s) - \int_0^1 \kappa(s,t)x(t)\, dt = f(s), \quad s \in [0,1],$$

where

$$\kappa(s,t) = \begin{cases} s(1-t), & 0 \le s \le t, \\ (1-s)t, & t \le s \le 1, \end{cases}$$

and the right-hand side $f(s)$ *is chosen such that the exact solution is* $\varphi(s) = s^{9/2}$.

Table 2.4.3: Comparison of convergence rates in collocation variants for Example 4.

n	$\left\|\varphi - \varphi_n^C\right\|_\infty$	δ_C	$\left\|\varphi - \varphi_n^S\right\|_\infty$	δ_{IC}	$\left\|\varphi - \varphi_n^M\right\|_\infty$	δ_M	$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty$	δ_{IM}
4	3.83×10^{-2}		1.70×10^{-3}		2.73×10^{-4}		1.40×10^{-4}	
8	2.47×10^{-2}	0.63	5.57×10^{-4}	1.61	3.90×10^{-5}	2.80	9.05×10^{-6}	3.95
16	1.09×10^{-2}	1.18	1.29×10^{-4}	2.11	5.03×10^{-6}	2.96	5.69×10^{-7}	3.99
32	5.79×10^{-3}	0.91	3.40×10^{-5}	1.92	6.34×10^{-7}	2.99	3.56×10^{-8}	4.00
64	2.80×10^{-3}	1.05	8.32×10^{-6}	2.03	7.94×10^{-8}	3.00	2.23×10^{-9}	4.00
128	1.42×10^{-3}	0.98	2.10×10^{-6}	1.98	9.93×10^{-9}	3.00	1.39×10^{-10}	4.00

Example 5. *Cconsider the Fredholm integral equation of the second kind,*

$$x(s) - \int_0^1 \kappa(s,t)x(t)\,dt = \left(1 - \frac{1}{\pi^2}\right)\sin(\pi s), \quad s \in [0,1],$$

where

$$\kappa(s,t) = \begin{cases} s(1-t), & 0 \le s \le t, \\ (1-s)t, & t \le s \le 1, \end{cases}$$

and the exact solution is given by $\varphi(s) = \sin(\pi s)$.

Table 2.4.4: Comparison of convergence rates in collocation variants for Example 5.

n	$\left\|\varphi - \varphi_n^C\right\|_\infty$	δ_C	$\left\|\varphi - \varphi_n^S\right\|_\infty$	δ_{IC}	$\left\|\varphi - \varphi_n^M\right\|_\infty$	δ_M	$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty$	δ_{IM}
2	7.26×10^{-1}		1.05×10^{-2}		3.75×10^{-3}		2.62×10^{-3}	
4	3.85×10^{-1}	0.92	2.16×10^{-3}	2.29	8.82×10^{-4}	2.09	1.51×10^{-4}	4.11
8	1.95×10^{-1}	0.98	5.13×10^{-4}	2.07	1.23×10^{-4}	2.84	9.36×10^{-6}	4.01
16	9.80×10^{-2}	0.99	1.27×10^{-4}	2.02	1.58×10^{-5}	2.96	5.84×10^{-7}	4.00
32	4.91×10^{-2}	1.00	3.15×10^{-5}	2.00	1.99×10^{-6}	2.99	3.65×10^{-8}	4.00
64	2.45×10^{-2}	1.00	7.88×10^{-6}	2.00	2.50×10^{-7}	3.00	2.28×10^{-9}	4.00

From Tables 2.4.3 and 2.4.4, we observe that the computed orders of convergence δ_C, δ_{IC}, δ_M, and δ_{IM} are in excellent agreement with the corresponding theoretical predictions. In particular, the iterative variants of both collocation and modified collocation methods exhibit a significant improvement in the rate of convergence, thereby confirming the theoretical superconvergence behaviour established in the preceding analysis.

The numerical results for both smooth and Green's function-type kernels validate the theoretical convergence and superconvergence results, confirming the effectiveness of the proposed projection-based methods.

2.5 Conclusion

In this chapter, projection-based collocation methods constructed on uniform partitions with equidistant (non-Gaussian) interpolation points were analyzed for Fredholm integral equations of the second kind. The theoretical results show that even with this simple choice of collocation nodes, substantial improvement in accuracy can be achieved through modified projection and one-step iteration methods.

For smooth kernels $\kappa \in \mathcal{C}^{2r+2}(\Omega)$, the modified and iterated modified collocation methods attain the superconvergent orders $O(h^{4r+3})$ and $O(h^{4r+4})$, respectively, significantly improving upon the standard collocation rate $O(h^{2r+1})$. For Green's function-type kernels with piecewise regularity, the modified methods retain enhanced convergence, yielding $O(h^{2r+2})$ and $O(h^{2r+3})$ for $r \geq 1$, and $O(h^3)$ and $O(h^4)$ in the lowest-order case $r = 0$.

These results establish that the proposed projection framework is both robust and systematically higher-order, providing accurate and computationally efficient approximations without requiring special quadrature points. The next chapter extends these ideas to the analysis of Urysohn integral equations of the second kind.

Chapter 3

Urysohn Integral Equations with Smooth and Green's Kernels[1]

This chapter is devoted to the analytical study of projection-based numerical methods for the approximate solution of nonlinear Urysohn integral equations of the form $x - \mathcal{K}(x) = f$, where f is a given function and $\mathcal{K}$ is a Urysohn integral operator with both smooth and Green's function type kernel acting on $\mathcal{L}^\infty[0,1]$. The numerical framework considered here is based on interpolatory projections onto finite-dimensional approximation spaces $\mathcal{X}_n$, consisting of piecewise polynomials of even degree defined with respect to a uniform partition of the interval $[0,1]$. The projection operator is constructed using equally spaced collocation points on each subinterval. Within this framework, four numerical methods are investigated: the standard collocation method, the iterated collocation method, the modified collocation method, and the iterated modified collocation method. A detailed convergence analysis is carried out for each of these schemes, with particular emphasis on comparing their accuracy and convergence behaviour.

This chapter extends the results of Chapter 2 to nonlinear integral equations and therefore requires a more detailed and in-depth analysis. Comparable orders of convergence are established for Urysohn integral operators with both smooth kernels and Green's function type kernels. Finally, numerical experiments are presented to illustrate and validate the theoretical results developed in this chapter.

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3.1 Preliminaries

Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$ and $\Omega \times \mathbb{R} = [0,1] \times [0,1] \times \mathbb{R}$. Let $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the non-linear integral operator $\mathcal{K}$ from $\mathcal{L}^\infty[0,1]$ to $\mathcal{C}[0,1]$ as

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\, dt, \quad s \in [0,1], \quad x \in \mathcal{X}. \tag{3.1.1}$$

Such an operator is called as Urysohn integral operator. Since the kernel κ is continuous, $\mathcal{K}$ is compact operator on $\mathcal{X}$. For given f and κ, we consider the following Urysohn integral equation

$$x - \mathcal{K}(x) = f. \tag{3.1.2}$$

The Fréchet derivative of $\mathcal{K}$ at $y \in \mathcal{X}$ is given by

$$(\mathcal{K}'(x)y)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))y(t)\, dt, \quad s \in [0,1],$$

where $\frac{\partial \kappa}{\partial u}(s,t,x(t))$ represents the partial derivative of κ with respect to its third component u at $x(t)$. Assuming that the equation (3.1.2) possesses a solution φ and, 1 is not an eigenvalue of the compact linear operator $\mathcal{K}'(\varphi)$. Then, φ is an isolated solution of (3.1.2). See [42, 44]. Note that, $\varphi \in \mathcal{C}^\alpha[0,1]$ whenever $f \in \mathcal{C}^\alpha[0,1]$ for any positive integer α. (See [10, Corollory 3.2]). Solving the equation (3.1.2) exactly, is often challenging, so we focus on finding approximate solutions. In collocation method, we employ a sequence of finite-rank interplatory projections that converge pointwise to the identity operator to approximate the operator $\mathcal{K}$. This method has been extensively studied in [42, 44, 43].

Let $\mathcal{X}_n$ denotes a sequence of finite-dimensional subspaces approximating $\mathcal{X}$, and let $\{\mathcal{Q}_n\}$ represents a sequence of interpolory projections from $\mathcal{X}$ to $\mathcal{X}_n$. In the collocation method, equation (3.1.2) is approximated by

$$\varphi_n^C - \mathcal{Q}_n\mathcal{K}(\varphi_n^C) = \mathcal{Q}_n f.$$

For linear integral equations, Sloan [83] introduced the iterated collocation solution, obtained through a single iteration of the collocation solution. This approach was further opted by Atkinson-Potra [10] for Urysohn integral equations. The iterated collocation solution is defined by

$$\varphi_n^S = \mathcal{K}(\varphi_n^C) + f.$$

Additionally, there exists an enhanced approach known as the modified collocation method, proposed by Grammont-Kulkarni [32] as

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f, \quad \text{with} \;\; \mathcal{K}_n^M(x) = \mathcal{Q}_n\mathcal{K}(x) + \mathcal{K}(\mathcal{Q}_n x) - \mathcal{Q}_n\mathcal{K}(\mathcal{Q}_n x).$$

This approach generalizes the modified collocation method for the linear case, as proposed by Kulkarni [48]. The solution obtained by applying a single iteration in φ_n^M is termed as the iterated modified collocation solution which is defined by

$$\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f.$$

For an integer $r \geq 0$, let $\mathcal{X}_n$ be the space of piecewise polynomials of even degree $\leq 2r$ on a uniform partition of $[0,1]$ with mesh length h, and let $\mathcal{Q}_n$ denote the interpolatory projection defined using $2r+1$ Gauss points (that is, the zeros of the corresponding Legendre polynomial) per subinterval. For Urysohn integral equations with sufficiently smooth kernels, the convergence of collocation and iterated collocation methods was analyzed by Atkinson-Potra [10]. The modified collocation method and its iterated version were studied by Grammont et al. [33], while analogous results for Green's function type kernels were obtained in [10, 57]. In all cases, iteration leads to an improvement in the order of convergence.

When the collocation points are not necessarily Gauss points, both collocation and iterated collocation methods converge with same order $O(h^{2r+1})$, see Atkinson-Potra [10]; whereas the modified and iterated modified collocation methods achieve the higher order $O(h^{4r+2})$, see Grammont et al. [33]. These results indicate that the choice of Gauss points as collocation points plays a crucial role in enhancing the convergence rates through iterative procedures.

In this thesis, we adopt a different choice of the approximating space and the collocation points from those considered in the aforementioned works. Accordingly, we briefly describe below the interpolatory projection employed throughout this chapter.

3.1.1 Interpolatory Projection

Let $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be the uniform partition of $[0,1]$, where

$$t_j = \frac{j}{n} = jh, \quad j = 0, 1, \ldots, n,$$

and $h = \dfrac{1}{n}$ denotes the mesh size. We write $\Delta_j = [t_{j-1}, t_j]$ for $j = 1, 2, \ldots, n$.
For a fixed integer $r \geq 0$, define the finite-dimensional approximating space

$$\mathcal{X}_n = \Big\{ x \in \mathcal{L}^\infty[0,1] : x|_{\Delta_j} \text{ is a polynomial of even degree } \leq 2r, \; j = 1, \ldots, n \Big\}.$$

On each subinterval Δ_j, we choose $2r+1$ interpolation points. For $r \geq 1$, these points are taken as

$$\tau_j^i = t_{j-1} + \frac{ih}{2r}, \quad i = 0, 1, \ldots, 2r,$$

while for $r = 0$ we use the midpoint

$$\tau_j = \frac{t_{j-1} + t_j}{2}.$$

We define the interpolatory operator $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ by prescribing the interpolation conditions

$$\mathcal{Q}_n x(\tau_j^i) = x(\tau_j^i), \quad i = 0, 1, \ldots, 2r, \quad j = 1, 2, \ldots, n. \tag{3.1.3}$$

Let $x \in \mathcal{C}[0,1]$. The $(2r+1)$-th divided difference of x at the points $\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}$ is given by

$$[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}]x = \frac{[\tau_j^1, \ldots, \tau_j^{2r}]x - [\tau_j^0, \ldots, \tau_j^{2r-1}]x}{\tau_j^{2r} - \tau_j^0}.$$

Define the nodal polynomial

$$\Psi_j(t) = \prod_{i=0}^{2r}(t - \tau_j^i), \quad j = 1, 2, \ldots, n.$$

Since $\mathcal{Q}_{n,j}x$ (the restriction of $\mathcal{Q}_n x$ to Δ_j) interpolates x at the points $\tau_j^0, \ldots, \tau_j^{2r}$, the interpolation error admits the representation

$$x(t) - \mathcal{Q}_{n,j}x(t) = \Psi_j(t)\,[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t]x, \quad t \in \Delta_j.$$

If $x \in \mathcal{C}^{2r+1}(\Delta_j)$, then standard interpolation theory yields

$$\|(\mathcal{I} - \mathcal{Q}_{n,j})x\|_{\Delta_j,\infty} \leq C_1 \left\|x^{(2r+1)}\right\|_{\Delta_j,\infty} h^{2r+1}, \tag{3.1.4}$$

where $C_1 > 0$ is a constant independent of h (see, [13] or Chapter 1).

The operator $\mathcal{Q}_n$ is initially defined on $\mathcal{C}[0,1]$ and, if necessary, extended to $\mathcal{L}^\infty[0,1]$ following the approach of Atkinson et al. [8]. In this setting, $\mathcal{Q}_n$ is a projection from $\mathcal{L}^\infty[0,1]$ onto $\mathcal{X}_n$ and satisfies

$$\mathcal{Q}_n x \to x, \quad x \in \mathcal{C}[0,1],$$

as $n \to \infty$. Moreover, by combining the local estimate (3.1.4) over all subintervals, it follows that for any $x \in \mathcal{C}^{2r+1}[0,1]$,

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq C_1 \left\|x^{(2r+1)}\right\|_\infty h^{2r+1}. \tag{3.1.5}$$

More generally, let $\alpha \geq 0$ be an integer and set $\beta = \min\{\alpha, 2r+1\}$. Then, for any $x \in \mathcal{C}^{\alpha}[0,1]$,

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_{\infty} \leq C_1 \big\|x^{(\beta)}\big\|_{\infty} h^{\beta}.$$

In Chapter 2, we analyzed the orders of convergence of approximate solutions of Fredholm integral equations with smooth and Green's function type kernels obtained by the collocation method, the iterated collocation method, and their modified variants. In that chapter, the space $\mathcal{X}_n$ was taken as the space of piecewise polynomials of even degree $\leq 2r$ with respect to the same uniform partition, and the projection $\mathcal{Q}_n$ was defined by interpolation at $2r+1$ equidistant points (including the endpoints) in each subinterval $[t_{j-1}, t_j]$ discussed above. Although these interpolation points are not Gauss points, we showed that iterative refinement still improves the orders of convergence for both the standard and modified collocation methods.

In the present chapter, we extend this theory to Urysohn integral equations with both smooth kernels and nonsmooth kernels of Green's function type. Specifically, we construct approximate solutions of (3.1.2) using collocation methods, modified collocation methods, and their iterated variants, and establish rigorous convergence results for each case. The corresponding orders of convergence are derived under appropriate regularity assumptions on the kernel and the right-hand side.

This chapter is organized as follows. In Section 3.2, we analyze the convergence properties of the proposed collocation-based methods for Urysohn integral equations with smooth kernels. Section 3.3 is devoted to the convergence analysis for integral equations involving non-smooth kernels of Green's function type. Numerical experiments that support and illustrate the theoretical results are presented in Section 3.4. Finally, concluding remarks and possible directions for future research are summarized in Section 3.5.

3.2 Convergence Analysis for Smooth Kernels

Let $\mathcal{X} = \mathcal{L}^{\infty}[0,1]$, and consider the Urysohn integral operator (3.1.1) as

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where the kernel $\kappa(s,t,u)$ is a real valued continuous function on $\Omega \times \mathbb{R}$. For an integer $\alpha \geq 0$, let $\mathcal{C}^{\alpha}[0,1]$ represent the space of all real-valued functions on $[0,1]$ which are α-times continuously differentiable, equipped with the norm

$$\|x\|_{\alpha,\infty} = \max_{0 \leq i \leq \alpha} \big\|x^{(i)}\big\|_{\infty} = \max_{0 \leq i \leq \alpha} \left\{ \sup_{0 \leq t \leq 1} |x^{(i)}(t)| \right\}.$$

where $x^{(i)}$ is the i-th derivative of the function x. Sometimes, for $x \in \mathcal{C}^\alpha[0,1]$, we define

$$\|x\|_{\alpha,\infty} = \sum_{i=0}^{\alpha} \|x^{(i)}\|_\infty.$$

Assume that

$$\kappa \in \mathcal{C}^\alpha(\Omega \times \mathbb{R}) \quad \text{and} \quad \frac{\partial \kappa}{\partial u} \in \mathcal{C}^{\alpha+p}(\Omega \times \mathbb{R}), \quad p \geq 1.$$

Then $\mathcal{K}$ is a compact operator from $\mathcal{L}^\infty[0,1]$ to $\mathcal{C}^\alpha[0,1]$. As in Section 3.1, we assume that (3.1.2) has a unique solution φ. We also assume that $f \in \mathcal{C}^\alpha[0,1]$. Then, since

$$\varphi - \mathcal{K}(\varphi) = f,$$

the solution $\varphi \in \mathcal{C}^\alpha[0,1]$. As φ denote the unique solution of equation (3.1.2) and define

$$q(s,t,u) = \frac{\partial \kappa}{\partial u}(s,t,u), \qquad \text{for all } (s,t,u) \in \Omega \times \mathbb{R};$$

and

$$q_*(s,t) = q(s,t,\varphi(t)).$$

For j,k non-negative integers, we set

$$D^{(j,k)} q_*(s,t) = \frac{\partial^{j+k} q_*}{\partial s^j \partial t^k}(s,t),$$

and define the constant as

$$C_2 = \max_{0 \leq j+k \leq 2r+3} \left\{ \sup_{0 \leq s,t \leq 1} \left| D^{(j,k)} q_*(s,t) \right| \right\}.$$

For $\delta_0 > 0$, let $\mathcal{B}(\varphi, \delta_0) = \{x \in \mathcal{L}^\infty[0,1] : \|x - \varphi\|_\infty < \delta_0\}$ be a neighborhood of the exact solution. Since $\dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{\alpha+p}$ with $p \geq 1$, the Fréchet derivative of $\mathcal{K}$ at $y \in \mathcal{L}^\infty[0,1]$ exists and is given by

$$(\mathcal{K}'(x)y)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))\, y(t)\, dt = \int_0^1 q_*(s,t) y(t)\ dt, \quad s \in [0,1].$$

Moreover, $\mathcal{K}'(x)$ defines a compact linear operator from $\mathcal{L}^\infty[0,1]$ into $\mathcal{C}^\alpha[0,1]$ for each $x \in \mathcal{B}(\varphi, \delta_0)$. The additional regularity $p \geq 1$ guarantees that $\mathcal{K}'$ is locally Lipschitz continuous in $\mathcal{B}(\varphi, \delta_0)$, that is, there exists a constant $\gamma > 0$ such that

$$\|\mathcal{K}'(\varphi) - \mathcal{K}'(x)\| \leq \gamma \|\varphi - x\|_\infty, \quad x \in \mathcal{B}(\varphi, \delta_0). \tag{3.2.1}$$

For an integer $r \geq 0$, whenever $\mathcal{X}_n$ be the space of piecewise polynomials of degree $\leq 2r$ associated with the same uniform partition of $[0,1]$ and mesh length $h = \dfrac{1}{n}$. In each subinterval $[t_{j-1}, t_j]$, choose $2r+1$ collocation points as Gauss points, and let $\mathcal{Q}_n : \mathcal{X} \to \mathcal{X}_n$ be the interpolatory projection at the collocation points. Then, the following orders of convergence are proved in Atkinson-Potra [10] for sufficiently smooth kernel:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{4r+2}\big)\,.$$

Subsequently, Grammont et al. [33] analyzed the modified collocation method and its iterated variant and proved the improved convergence estimates

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{6r+3}\big)\,, \quad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{8r+4}\big)\,.$$

As discussed in Section 3.1, when the interpolatory projection is defined using $2r + 1$ distinct collocation points in each subinterval that are not necessarily Gauss points, Atkinson-Potra [10] showed that, for sufficiently smooth kernels, both the collocation and the iterated collocation methods converge with order $O(h^{2r+1})$. This behaviour can be illustrated by the following example.

Example 6. *Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$, and let $\mathcal{X}_n$ denote the space of piecewise constant functions with respect to the uniform partition of $[0,1]$ given by*

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \; j = 0, 1, \ldots, n.$$

Define the interpolatory operator $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ by interpolation at the collocation points

$$\tau_j = \frac{3j-2}{3n}, \quad j = 1, 2, \ldots, n.$$

Using the Hahn–Banach extension theorem, $\mathcal{Q}_n$ can be extended to $\mathcal{L}^\infty[0,1]$, and hence $\mathcal{Q}_n$ is a projection. Consider the Urysohn integral equation

$$x - \mathcal{K}(x) = f,$$

where $f(s) = s$, $s \in [0,1]$, and

$$\mathcal{K}(x)(s) = \left(\int_0^1 x(t)\,dt + 1\right) s, \quad s \in [0,1].$$

The exact solution of this equation is given by $\varphi(s) = 4s$, $s \in [0,1]$. The operator $\mathcal{K}$ is compact and Fréchet differentiable, with Fréchet derivative

$$\big(\mathcal{K}'(\varphi)y\big)(s) = \left(\int_0^1 y(t)\,dt\right) s, \quad s \in [0,1], \quad y \in \mathcal{X}.$$

Moreover, 1 *is not an eigenvalue of* $\mathcal{K}'(\varphi)$.

For $s \in [t_{j-1}, t_j]$, *we have*

$$(\mathcal{I} - \mathcal{Q}_n)\varphi(s) = 4s - \frac{4(3j-2)}{3n}.$$

A direct computation yields

$$\int_0^1 (\mathcal{I} - \mathcal{Q}_n)\varphi(t)\,dt = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left(4t - \frac{4(3j-2)}{3n}\right) dt = \frac{2}{3n}.$$

Consequently,

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = \max_{s\in[0,1]} \left| \left(\int_0^1 (\mathcal{I} - \mathcal{Q}_n)\varphi(t)\,dt \right) s \right| = \frac{2}{3n},$$

and

$$\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = \max_{1\le j\le n} \left\{ \max_{s\in[t_{j-1},t_j]} |(\mathcal{I} - \mathcal{Q}_n)\varphi(s)| \right\} \le \frac{8}{3n}.$$

In general, the iterated collocation solution is superconvergent (see [13]) if and only if

$$\frac{\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty}{\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty} \longrightarrow 0 \quad \textit{as } n \to \infty.$$

In the present case, this condition is not satisfied, since

$$\frac{\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty}{\|(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty} \ge \frac{1}{4}.$$

Thus, the iterated collocation method does not exhibit superconvergence for this choice of interpolatory projection.

In the same setting, Grammont et al. [33] further showed that the modified collocation method and its iterated variant yield an improved convergence rate of order $O(h^{4r+2})$. We observe that selecting Gauss points as collocation points improves the convergence orders through iterative refinement.

In our analysis, we employ an approximating space $\mathcal{X}_n$ consisting of piecewise polynomials of even degree $\le 2r$, together with equidistant collocation points. Recall that $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denotes the interpolatory projection defined at $(2r+1)n$ equidistant points in $[0,1]$ defined by (3.1.3). As shown in Chapter 1, the operator $\mathcal{Q}_n$ admits a bounded extension to a projection from $\mathcal{L}^\infty[0,1]$ onto $\mathcal{X}_n$.

In particular, $\mathcal{Q}_n x \to x$ uniformly for all $x \in \mathcal{C}[0,1]$. Moreover, if $x \in \mathcal{C}^{2r+1}[0,1]$, then the interpolation error satisfies

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \le C_1 \left\|x^{(2r+1)}\right\|_\infty h^{2r+1},$$

where $C_1 > 0$ is a constant independent of n.

We briefly recall the approximation methods used to compute numerical solutions of the Urysohn integral equation (3.1.2), which are summarized below.

The collocation approximation of (3.1.2) is defined by

$$\varphi_n^C - \mathcal{Q}_n \mathcal{K}(\varphi_n^C) = \mathcal{Q}_n f. \tag{3.2.2}$$

The corresponding iterated collocation solution is given by

$$\varphi_n^S = \mathcal{K}(\varphi_n^C) + f. \tag{3.2.3}$$

Also, the modified collocation method is defined by

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f, \tag{3.2.4}$$

where the modified operator $\mathcal{K}_n^M$ is given by

$$\mathcal{K}_n^M(x) = \mathcal{Q}_n \mathcal{K}(x) + \mathcal{K}(\mathcal{Q}_n x) - \mathcal{Q}_n \mathcal{K}(\mathcal{Q}_n x).$$

The associated iterated modified collocation solution is defined as

$$\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f. \tag{3.2.5}$$

This approach is particularly effective in enhancing the convergence rate while maintaining computational efficiency.

3.2.1 Collocation and Iterated Collocation Methods

Note that the Fréchet derivative $\mathcal{K}'(\varphi)$ is a compact linear operator on $\mathcal{X}$. Since 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$, the operator $\mathcal{I} - \mathcal{K}'(\varphi)$ is boundedly invertible on $\mathcal{X}$. Moreover, classical results of Krasnoselskii [42], Krasnoselskii-Zabreiko [44], and Atkinson-Potra [10] ensure that, for all sufficiently large n, the collocation equation (3.2.2) admits a unique solution φ_n^C in a neighbourhood of φ, and that $\varphi_n^C \to \varphi$ as $n \to \infty$. Moreover, the errors of the collocation and iterated collocation approximations satisfy the estimates

$$\left\|\varphi - \varphi_n^C\right\|_\infty \leq C_3 \left\|(\mathcal{I} - \mathcal{Q}_n)\varphi\right\|_\infty \tag{3.2.6}$$

and

$$\left\|\varphi - \varphi_n^S\right\|_\infty \leq C_4 \left\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\right\|_\infty, \tag{3.2.7}$$

where C_3 and C_4 are positive constants independent of n. See Chapter 1. The following result provides an error bound for the composition of the linear operator $\mathcal{K}'(\varphi)$ with the interpolation operator $\mathcal{Q}_n$.

Proposition 23. *Assume that q_* is continuously differentiable with respect to t and that $x \in \mathcal{C}^{2r+2}[0,1]$. Then*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_2 \|x\|_{2r+2,\infty} h^{2r+2}.$$

Proof. We recall that

$$(\mathcal{K}'(\varphi)x)(s) = \int_0^1 q_*(s,t)\, x(t)\, dt, \quad q_*(s,t) = \frac{\partial \kappa}{\partial u}(s,t,\varphi(t)).$$

For $s \in [0,1]$, we write

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) &= \int_0^1 q_*(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\, dt \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} q_*(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})x(t)\, dt.\end{aligned}$$

On each subinterval $[t_{j-1}, t_j]$, the interpolation remainder satisfies

$$(\mathcal{I}-\mathcal{Q}_{n,j})x(t) = \Psi_j(t)\, [\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, t]x, \quad t \in [t_{j-1}, t_j],$$

where $\Psi_j(t) = \prod_{i=0}^{2r}(t-\tau_j^i)$. Hence,

$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} q_*(s,t)\Psi_j(t)[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, t]x\, dt.$$

For fixed s, define

$$f_j^s(t) = q_*(s,t)[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, t]x, \quad t \in [t_{j-1}, t_j].$$

Then

$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} f_j^s(t)\Psi_j(t)\, dt. \tag{3.2.8}$$

Define

$$\omega_j(t) = \int_{t_{j-1}}^{t} \Psi_j(\xi)\, d\xi.$$

Then $\omega_j'(t) = \Psi_j(t)$, with $\omega_j(t_{j-1}) = \omega_j(t_j) = 0$. Using integration by parts in (3.2.8), we obtain

$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) = -\sum_{j=1}^n \int_{t_{j-1}}^{t_j} (f_j^s)'(t)\, \omega_j(t)\, dt.$$

Since

$$(f_j^s)'(t) = q_*(s,t)[\tau_j^0, \dots, \tau_j^{2r}, t, t]x + D^{(0,1)}q_*(s,t)[\tau_j^0, \dots, \tau_j^{2r}, t]x,$$

and $x \in \mathcal{C}^{2r+2}[0,1]$, standard divided difference estimates (Hildebrand [36]) yield

$$|(f_j^s)'(t)| \leq 2C_2 \, \|x\|_{2r+2,\infty}, \quad t \in [t_{j-1}, t_j].$$

Moreover, $\|\Psi_j\|_\infty = O(h^{2r+1})$ and $\|\omega_j\|_\infty = O(h^{2r+2})$. Therefore,

$$\left|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x(s)\right| \leq 2C_2 \, \|x\|_{2r+2,\infty} \, h^{2r+2}.$$

Taking the supremum over $s \in [0,1]$ completes the proof. □

The above estimate will be now used to determine the convergence orders of the collocation and iterated collocation methodss. In particular, the convergence order of the collocation solution of (3.2.2) is obtained by combining the interpolation error estimate (3.1.5) with the bound (3.2.6). Similarly, the convergence order of the iterated collocation solution defined by (3.2.3) follows from Proposition 23 together with the estimate (3.2.7). As a result, we obtain

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big), \quad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+2}\big).$$

These results demonstrate that the iterated collocation method achieves a higher order of convergence than the standard collocation method, despite the fact that the interpolation points are not chosen to be Gauss points.

3.2.2 Modified Collocation Method

We first present following lemmas that are essential for the proof of Theorem 24, where the convergence orders of the modified collocation solution are established.

Lemma 3.2.1. *Let $q_* \in \mathcal{C}^{2r+1}(\Omega)$ and that $x \in \mathcal{C}^{2r+2}[0,1]$. Then*

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq 2C_1C_2\|x\|_{2r+2,\infty} \, h^{4r+3}.$$

Proof. Since $q_*(s,t) \in \mathcal{C}^{2r+1}(\Omega)$, that is, $\dfrac{\partial \kappa}{\partial u}(s,t,\varphi(t)) \in \mathcal{C}^{2r+1}(\Omega)$, and since $x \in \mathcal{C}^{2r+2}[0,1]$, it follows that

$$\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x \in \mathcal{C}^{2r+1}[0,1].$$

Using the interpolation error estimate (3.1.5), we obtain

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq C_1 \left\| \big(\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x\big)^{(2r+1)} \right\|_\infty h^{2r+1}. \tag{3.2.9}$$

Consider
$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) = \int_0^1 q_*(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt.$$
Since $q_* \in \mathcal{C}^{2r+1}(\Omega)$, we obtain
$$\begin{aligned}\big(\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\big)^{(2r+1)}(s) &= \int_0^1 D^{(2r+1,0)}q_*(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt\\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} D^{(2r+1,0)}q_*(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})x(t)\,dt\\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} D^{(2r+1,0)}q_*(s,t)\Psi_j(t)[\tau_j^0,\ldots,\tau_j^{2r},t]x\,dt.\end{aligned}$$

For fixed s, define
$$g_j^s(t) = D^{(2r+1,0)}q_*(s,t)[\tau_j^0,\ldots,\tau_j^{2r},t]x, \quad t\in[t_{j-1},t_j].$$

Then
$$\big(\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\big)^{(2r+1)}(s) = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\,dt.$$
Since $q_* \in \mathcal{C}^{2r+1}(\Omega)$ and $x\in\mathcal{C}^{2r+1}[0,1]$, standard divided-difference estimates imply that, for all $s\in[0,1]$ and $t\in[t_{j-1},t_j]$,
$$\big|(g_j^s)'(t)\big| \le 2C_2\|x\|_{2r+2,\infty}.$$

Let c_j denote the midpoint of $\Delta_j = [t_{j-1},t_j]$. By the mean-value form of Taylor's theorem, for each $t\in\Delta_j$ there exists $\xi_t\in\Delta_j$ such that
$$g_j^s(t) = g_j^s(c_j) + (t-c_j)(g_j^s)'(\xi_t).$$

Using the antisymmetry of Ψ_j about c_j (Lemma 1.3.1, Chapter 1), we obtain
$$\int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\,dt = \int_{t_{j-1}}^{t_j}(t-c_j)(g_j^s)'(\xi_t)\Psi_j(t)\,dt.$$

Hence,
$$\left|\int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\,dt\right| \le 2C_2\|x\|_{2r+2,\infty}\int_{t_{j-1}}^{t_j}|t-c_j|\,|\Psi_j(t)|\,dt.$$
Since $|t-c_j| \le \dfrac{h}{2}$ on Δ_j and $\int_{t_{j-1}}^{t_j}|\Psi_j(t)|\,dt = O(h^{2r+2})$, each subinterval contributes $O(h^{2r+3})$. Summing over $j=1,\ldots,n$ yields
$$\left\|\big(\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\big)^{(2r+1)}\right\|_\infty \le 2C_2\|x\|_{2r+2,\infty}\,h^{2r+2}.$$

Substituting into (3.2.9) gives

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \le 2C_1C_2\|x\|_{2r+2,\infty}\, h^{4r+3},$$

which completes the proof. □

Lemma 3.2.2. *Assume that the Urysohn integral equation* (3.1.2) *admits a solution* $\varphi \in \mathcal{C}^{2r+1}[0,1]$*, and that*

$$\frac{\partial\kappa}{\partial u} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R}).$$

Then

$$\big\|(\mathcal{I} - \mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\big]\big\|_\infty = O\big(h^{6r+3}\big)\,.$$

Proof. Since by assumption

$$\frac{\partial\kappa}{\partial u} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R}),$$

it follows in particular that $\dfrac{\partial\kappa}{\partial u}$ is continuously differentiable with respect to u and possesses continuous mixed derivatives with respect to s up to order $2r+1$. Consequently, for $y_1, y_2 \in \mathcal{L}^\infty[0,1]$, the second Fréchet derivative of $\mathcal{K}$ at φ exists and is given by

$$\mathcal{K}''(\varphi)(y_1, y_2)(s) = \int_0^1 \frac{\partial^2\kappa}{\partial u^2}(s,t,\varphi(t))\, y_1(t)y_2(t)\,dt.$$

The norm of the bilinear operator $\mathcal{K}''(\varphi)$ is defined by

$$\|\mathcal{K}''(\varphi)\| = \sup_{\|y_1\|_\infty \le 1,\ \|y_2\|_\infty \le 1} \|\mathcal{K}''(\varphi)(y_1, y_2)\|_\infty.$$

Let $v \in \mathcal{B}(\varphi, \delta_0)$. By generalised Taylor's theorem with integral remainder, we may write

$$\begin{aligned}\mathcal{K}(\varphi + v)(s) - \mathcal{K}(\varphi)(s) - \mathcal{K}'(\varphi)v(s) &= \int_0^1 (1-\theta)\,\mathcal{K}''(\varphi + \theta v)(v,v)(s)\,d\theta \\ &=: (Rv)(s), \quad s \in [0,1].\end{aligned}$$

Note that

$$\mathcal{K}''(\varphi + \theta v)(v,v)(s) = \int_0^1 \frac{\partial^2\kappa}{\partial u^2}(s,t,\varphi(t) + \theta v(t))\, v^2(t)\,dt.$$

Since $\dfrac{\partial\kappa}{\partial u} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, it follows that $\mathcal{K}''(\varphi + \theta v)(v,v)(s)$, and hence Rv, belong to $\mathcal{C}^{2r+1}[0,1]$. We write

$$\left(\mathcal{K}''(\varphi + \theta v)(v,v)\right)^{(2r+1)}(s) = \int_0^1 \frac{\partial^{2r+1}}{\partial s^{2r+1}}\left(\frac{\partial^2\kappa}{\partial u^2}\,(s,t,\varphi(t) + \theta v(t))\right) v^2(t)\,dt,$$

and

$$(Rv)^{(2r+1)}(s) = \int_0^1 (1-\theta)\left(\mathcal{K}''(\varphi+\theta v)(v,v)\right)^{(2r+1)}(s)\,d\theta.$$

Let

$$C_5 = \max_{\substack{s,t\in[0,1]\\ |u|\le\|\varphi\|_\infty+\delta_0}} \left|\frac{\partial^{2r+1}}{\partial s^{2r+1}}\left(\frac{\partial^2\kappa}{\partial u^2}(s,t,u)\right)\right|.$$

Then, for all $\theta\in[0,1]$,

$$\left\|\left(\mathcal{K}''(\varphi+\theta v)(v,v)\right)^{(2r+1)}\right\|_\infty \le C_5\|v\|_\infty^2, \quad \left\|(Rv)^{(2r+1)}\right\|_\infty \le \frac{C_5}{2}\|v\|_\infty^2.$$

Using the interpolation estimate (3.1.5), we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)Rv\|_\infty \le C_1\left\|(Rv)^{(2r+1)}\right\|_\infty h^{2r+1} \le \frac{C_1C_5}{2}\|v\|_\infty^2 h^{2r+1}.$$

Since $\mathcal{Q}_n x\to x$ for all $x\in\mathcal{C}[0,1]$, it follows that for n sufficiently large, $v=\mathcal{Q}_n\varphi-\varphi\in\mathcal{B}(\varphi,\delta_0)$. Hence,

$$\|(\mathcal{I}-\mathcal{Q}_n)R(\mathcal{Q}_n\varphi-\varphi)\|_\infty \le \frac{C_1C_5}{2}\|\mathcal{Q}_n\varphi-\varphi\|_\infty^2 h^{2r+1}.$$

Finally, since $\varphi\in\mathcal{C}^{2r+1}[0,1]$, again by (3.1.5),

$$\|\mathcal{Q}_n\varphi-\varphi\|_\infty \le C_1\left\|\varphi^{(2r+1)}\right\|_\infty h^{2r+1}.$$

Therefore,

$$\begin{aligned}\|(\mathcal{I}-\mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)]\|_\infty &= \|(\mathcal{I}-\mathcal{Q}_n)R(\mathcal{Q}_n\varphi-\varphi)\|_\infty\\ &\le \frac{(C_1)^3C_5}{2}\left\|\varphi^{(2r+1)}\right\|_\infty^2 h^{6r+3},\end{aligned}$$

which completes the proof. □

The above proof is adapted from the argument used in Lemma 2.1 of Grammont et al. [33]. On the basis of the crucial results established in the preceding lemmas, we are now in a position to state the main convergence result for the modified collocation method. In particular, the assumptions required for the application of Theorem 7 are satisfied in the present setting with $f\ne 0$. Consequently, there exists a neighbourhood $\mathcal{B}(\varphi,\delta_0)$ of the exact solution φ such that, for all sufficiently large n, the modified collocation equation (3.2.4) admits a unique solution $\varphi_n^M\in\mathcal{B}(\varphi,\delta_0)$. We now state the corresponding error estimate for φ_n^M.

Theorem 24. *Assume that the Urysohn integral equation* (3.1.2) *admits a unique solution* φ, *and suppose that* $\kappa \in \mathcal{C}(\Omega \times \mathbb{R}), \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$ *and* $f \in \mathcal{C}^{2r+2}[0,1]$. *Assume further that* 1 *is not an eigenvalue of the Fréchet derivative* $\mathcal{K}'(\varphi)$. *Let* $\mathcal{X}_n$ *be the space of piecewise polynomials of even degree* $\leq 2r$ *with respect to the partition* (2.2), *and let* $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ *be the associated interpolatory projection defined by* (3.1.3). *Let* φ_n^M *denote the unique solution of the modified collocation equation* (3.2.4) *in a neighbourhood* $B(\varphi, \delta_0)$ *of* φ. *Then the modified collocation approximation satisfies the superconvergence estimate*

$$\left\|\varphi_n^M - \varphi\right\|_\infty = O\big(h^{4r+3}\big)\,,$$

as $n \to \infty$.

Proof. By assumption, the operator $\mathcal{I} - \mathcal{K}'(\varphi)$ is boundedly invertible. It follows from Lemma 1.4.3 that, for all sufficiently large n, the operator $\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)$ is also invertible. Moreover, its inverse is uniformly bounded, satisfying

$$\left\|\left[\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)\right]^{-1}\right\| \leq 2\left\|(\mathcal{I} - \mathcal{K}'(\varphi))^{-1}\right\|.$$

An application of Theorem 7 then yields

$$\left\|\varphi_n^M - \varphi\right\|_\infty \leq 2\,\alpha_n,$$

where

$$\begin{aligned}\alpha_n &= \left\|\left[\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)\right]^{-1}\left[\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\right]\right\| \\ &= \left\|\left[\mathcal{I} - (\mathcal{K}_n^M)'(\varphi)\right]^{-1}(\mathcal{I} - \mathcal{Q}_n)\big(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi)\big)\right\|_\infty.\end{aligned}$$

Combining the above estimates, we finally obtain

$$\left\|\varphi_n^M - \varphi\right\|_\infty \leq 4\left\|(I - \mathcal{K}'(\varphi))^{-1}\right\|\|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi))\|_\infty. \tag{3.2.10}$$

We write

$$\begin{aligned}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi)) = &-(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)] \\ &- (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi).\end{aligned}$$

By Lemma 3.2.2, we have

$$\|(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)]\|_\infty = O\big(h^{6r+3}\big)\,. \tag{3.2.11}$$

On the other hand, under the assumptions $f \in \mathcal{C}^{2r+2}[0,1]$ and $\dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, the unique solution $\varphi \in \mathcal{C}^{2r+2}[0,1]$. Then by Lemma 3.2.1, we have

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\|_\infty = \|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty = O\big(h^{4r+3}\big)\,.$$

From (3.2.10), (3.2.11) and the above estimate, we obtain

$$\left\|\varphi_n^M-\varphi\right\|_\infty = O\big(h^{4r+3}\big)\,.$$

□

3.2.3 Iterated Modified Collocation Method

In this subsection, we present the main result of the smooth kernel case. In particular, we show that the iterated modified collocation method attains a higher order of convergence than the modified collocation method. The proof of this result is carried out through a sequence of auxiliary lemmas and propositions, which collectively establish the main theorem.

Lemma 3.2.3. *Assume that $q_* \in \mathcal{C}^{2r+2}(\Omega)$ and that $x \in \mathcal{C}^{2r+2}[0,1]$. Then*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 4(C_2)^2\,\|x\|_{2r+2,\infty}\,h^{4r+4}.$$

Proof. Applying Proposition 23, we obtain

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_2\,\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty}\,h^{2r+2}. \qquad (3.2.12)$$

To estimate the derivative norm, observe that

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty} = \max_{0\le i\le 2r+2}\left\|\big(\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\big)^{(i)}\right\|_\infty.$$

Since $q_* \in \mathcal{C}^{2r+2}(\Omega)$, the proof proceeds exactly as in Lemma 3.2.1. In particular, one obtains

$$\left|\big(\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\big)^{(2r+2)}(s)\right| \le 2C_2\,\|x\|_{2r+2,\infty}\,h^{2r+2}, \quad s\in[0,1].$$

Hence,

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_{2r+2,\infty} \le 2C_2\,\|x\|_{2r+2,\infty}\,h^{2r+2}.$$

Substituting this estimate into (3.2.12) yields

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\,\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 4(C_2)^2\,\|x\|_{2r+2,\infty}\,h^{4r+4},$$

which completes the proof. □

Lemma 3.2.4. *Assume that* $\kappa, \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R}), \dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, *and* $f \in \mathcal{C}^{2r+2}[0,1]$. *Then*

$$\left\| \mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\big]\right\|_\infty = O\big(h^{6r+4}\big)\,.$$

Proof. Define

$$x_n = \mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi).$$

By definition of the Fréchet derivative,

$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x_n(s) = \int_0^1 q_*(s,t)(\mathcal{I}-\mathcal{Q}_n)x_n(t)\,dt, \quad s\in[0,1].$$

As $\kappa, \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega\times\mathbb{R}), \dfrac{\partial^2\kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega\times\mathbb{R})$, and $f\in\mathcal{C}^{2r+2}[0,1]$, then the exact solution satisfies $\varphi\in\mathcal{C}^{2r+2}[0,1]$. Moreover, the nonlinear remainder

$$x_n = \mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\in\mathcal{C}^{2r+2}[0,1].$$

Since q_* is continuously differentiable with respect to t and $x_n\in\mathcal{C}^{2r+2}[0,1]$, Proposition 23 is applicable with $x=x_n$, yielding

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x_n\|_\infty \le 2C_2\,\|x_n\|_{2r+2,\infty}\,h^{2r+2}. \tag{3.2.13}$$

From Lemma 3.2.2, we recall that

$$x_n = \mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi) = R(\mathcal{Q}_n\varphi-\varphi).$$

Moreover, by following Lemma 3.2.2, one can see that

$$\|x_n\|_{2r+2,\infty} = \|R(\mathcal{Q}_n\varphi-\varphi)\|_{2r+2,\infty} = \max_{0\le i\le 2r+2}\left\|\big(R(\mathcal{Q}_n\varphi-\varphi)\big)^{(i)}\right\|_\infty \le \frac{C_5}{2}\|\mathcal{Q}_n\varphi-\varphi\|_\infty^2.$$

Since $\varphi\in\mathcal{C}^{2r+2}[0,1]$, one can obtain

$$\|\mathcal{Q}_n\varphi-\varphi\| \le C_1\big\|\varphi^{(2r+1)}\big\|_\infty h^{2r+1}.$$

Thus,

$$\|x_n\|_{2r+2,\infty} \le \frac{(C_1)^2C_5}{2}\big\|\varphi^{(2r+1)}\big\|_\infty^2 h^{4r+2}. \tag{3.2.14}$$

Combining (3.2.13) and (3.2.14) together, we obtain

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x_n\|_\infty \le (C_1)^2C_2C_5\,\big\|\varphi^{(2r+1)}\big\|_\infty^2 h^{6r+4},$$

which proves the result. □

Lemma 3.2.5. *Assume that* $\kappa, \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R}), \dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, *and* $f \in \mathcal{C}^{2r+2}[0,1]$. *Then*

$$\left\|\mathcal{K}'(\varphi)\big(\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\big)\right\|_\infty = O\big(h^{4r+4}\big)\,.$$

Proof. By definition of the modified collocation operator,

$$\mathcal{K}_n^M(\varphi) = \mathcal{Q}_n\mathcal{K}(\varphi) + \mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{Q}_n\mathcal{K}(\mathcal{Q}_n\varphi).$$

Hence,

$$\begin{aligned}
\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi) &= (\mathcal{I} - \mathcal{Q}_n)\big(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi)\big) \\
&= -(\mathcal{I} - \mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\big] \\
&\quad - (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi).
\end{aligned}$$

Applying $\mathcal{K}'(\varphi)$ and taking norms, we obtain

$$\begin{aligned}
\left\|\mathcal{K}'(\varphi)\big(\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\big)\right\|_\infty &\le \left\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\big]\right\|_\infty \\
&\quad + \|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\|_\infty.
\end{aligned}$$

The first term is estimated directly by Lemma 3.2.4, yielding

$$\left\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\big]\right\|_\infty = O\big(h^{6r+4}\big)\,.$$

To estimate the second term, we observe that

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\|_\infty = \|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty$$

Since $\dfrac{\partial \kappa}{\partial u}(s,t,\varphi(t)) \in \mathcal{C}^{2r+2}(\Omega)$ and $\varphi \in \mathcal{C}^{2r+2}[0,1]$, Lemma 3.2.3 applies and we obtain

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\|_\infty = O\big(h^{4r+4}\big)\,.$$

Combining the above estimates yields

$$\left\|\mathcal{K}'(\varphi)\big(\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\big)\right\|_\infty = O\big(h^{4r+4}\big)\,,$$

which completes the proof. $\square$

Lemma 3.2.6. *Assume that* $\kappa, \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R}), \dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, *and* $f \in \mathcal{C}^{2r+2}[0,1]$. *Then*

$$\left\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)(\varphi - \varphi_n^M)\right\|_\infty = O\big(h^{6r+4}\big)\,.$$

Proof. By definition of the Fréchet derivative of the modified collocation operator,

$$(\mathcal{K}_n^M)'(\varphi) = \mathcal{Q}_n \mathcal{K}'(\varphi) + (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n.$$

Hence,

$$\begin{aligned}\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big) &= -\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)\\ &\quad + \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n.\end{aligned}$$

We decompose the second term as

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n &= \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\big(\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi)\big)\mathcal{Q}_n\\ &\quad + \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)\mathcal{Q}_n.\end{aligned}$$

By Kulkarni-Nelakanti [50, Theorem 4.2] as

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)\| = O\big(h^{4r+2}\big)\,.$$

The Lipschitz continuity of $\mathcal{K}'$ in (3.2.1) implies

$$\|\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi)\|_\infty \le \gamma\|\mathcal{Q}_n\varphi - \varphi\|_\infty \le \gamma C_1 \big\|\varphi^{(2r+1)}\big\|_\infty h^{2r+1},$$

and the uniform boundedness of $\mathcal{Q}_n$ then gives

$$\big\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\big(\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi)\big)\mathcal{Q}_n\big\|_\infty = O\big(h^{2r+1}\big)\,.$$

Combining the above bounds, we obtain

$$\big\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)\big\|_\infty = O\big(h^{2r+1}\big)\,.$$

Finally, from the convergence estimate for the modified collocation solution established in Theorem 24, we have

$$\big\|\varphi_n^M - \varphi\big\|_\infty = O\big(h^{4r+3}\big)\,.$$

Therefore,

$$\begin{aligned}\big\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)(\varphi - \varphi_n^M)\big\|_\infty &\le \big\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)\big\|_\infty \big\|\varphi - \varphi_n^M\big\|_\infty\\ &= O\big(h^{2r+1}\big)\cdot O\big(h^{4r+3}\big) = O\big(h^{6r+4}\big)\,,\end{aligned}$$

which cmpletes the proof. $\square$

The next theorem constitutes the principal result of this section. It quantifies the superconvergence achieved by the iterated modified collocation method when applied to Urysohn integral equations with smooth kernels.

Theorem 25. *Let $\mathcal{X}_n$ be the space of piecewise polynomials of even degree $\leq 2r$ with respect to the uniform partition of $[0,1]$, and let $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ be the interpolatory projection defined by* (3.1.3)*. Assume that $\kappa, \dfrac{\partial \kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R}), \dfrac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, and $f \in \mathcal{C}^{2r+2}[0,1]$. Further, assume that 1 is not an eigenvalue of $\mathcal{K}'(\varphi)$. Let φ be the unique solution of the Urysohn integral equation* (3.1.2)*, and let $\widetilde{\varphi}_n^M \in \mathcal{B}(\varphi, \delta_0)$ be the iterated modified collocation solution defined by* (3.2.5)*. Then*

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{4r+4}\big).$$

Proof. From the definitions of the exact and modified collocation equations, we have

$$\widetilde{\varphi}_n^M - \varphi = \mathcal{K}(\varphi_n^M) - \mathcal{K}(\varphi).$$

Applying the generalized Taylor expansion of $\mathcal{K}$ about φ,

$$\mathcal{K}(\varphi_n^M) - \mathcal{K}(\varphi) = \mathcal{K}'(\varphi)(\varphi_n^M - \varphi) + R(\varphi_n^M - \varphi), \tag{3.2.15}$$

where the remainder term is given by

$$\big(R(\varphi_n^M - \varphi)\big)(s) = \int_0^1 (1-\theta)\left(\mathcal{K}''\big(\varphi + \theta(\varphi_n^M - \varphi)\big)(\varphi_n^M - \varphi)^2\right)(s)\, d\theta, \quad s \in [0,1].$$

Since $\dfrac{\partial^2 \kappa}{\partial u^2}$ is continuous on $\Omega \times \mathbb{R}$. Then the operator $\mathcal{K}$ is twice Fréchet differentiable and the second Fréchet derivative of $\mathcal{K}$ is given by,

$$\left[\mathcal{K}''(\varphi + \theta(\varphi_n^M - \varphi))(\varphi_n^M - \varphi)^2\right](s) = \int_0^1 \frac{\partial^2 \kappa}{\partial u^2}(s,t,\varphi(t) + \theta(\varphi_n^M - \varphi)(t))(\varphi_n^M - \varphi)^2(t)\; dt.$$

Then

$$\left|\left[\mathcal{K}''(\varphi + \theta(\varphi_n^M - \varphi))(\varphi_n^M - \varphi)^2\right](s)\right| \leq \left(\sup_{\substack{s,t \in [0,1] \\ |u| \leq \|\varphi\|_\infty + \delta_0}} \left| \frac{\partial^2 \kappa}{\partial u^2}(s,t,u) \right| \right) \left\|\varphi_n^M - \varphi\right\|_\infty^2.$$

Thus,

$$\left\|\left[\mathcal{K}''(\varphi + \theta(\varphi_n^M - \varphi))(\varphi_n^M - \varphi)^2\right]\right\|_\infty \leq C_6 \left\|\varphi_n^M - \varphi\right\|_\infty^2,$$

where

$$C_6 = \sup_{\substack{s,t \in [0,1] \\ |u| \leq \|\varphi\|_\infty + \delta_0}} \left| \frac{\partial^2 \kappa}{\partial u^2}(s,t,u) \right|$$

is a constant. Consequently, we have

$$\left\|R(\varphi_n^M-\varphi)\right\|_\infty \leq \frac{C_6}{2}\left\|\varphi_n^M-\varphi\right\|_\infty^2.$$

Using (3.2.15), we arrive at the error estimate

$$\left\|\widetilde{\varphi}_n^M-\varphi\right\|_\infty \leq \left\|\mathcal{K}'(\varphi)(\varphi_n^M-\varphi)\right\|_\infty + C_6\left\|\varphi_n^M-\varphi\right\|_\infty^2. \tag{3.2.16}$$

From Theorem 24 we already know that

$$\left\|\varphi_n^M-\varphi\right\|_\infty = O\big(h^{4r+3}\big)\,.$$

As discussed in Grammont et al. [33], define the operator $B_n : \mathcal{X} \to \mathcal{X}$ by

$$B_n(x) = \varphi - (\mathcal{I}-\mathcal{K}'(\varphi))^{-1}\Big(\mathcal{K}'(\varphi) - \mathcal{K}'(\varphi)\varphi - \mathcal{K}_n^M(x) + \mathcal{K}'(\varphi)x\Big).$$

Then $B_n(x) = x$ if and only if

$$(\mathcal{I}-\mathcal{K}'(\varphi))x = (\mathcal{I}-\mathcal{K}'(\varphi))\varphi - \big(\mathcal{K}'(\varphi) - \mathcal{K}'(\varphi)\varphi - \mathcal{K}_n^M(x) + \mathcal{K}'(\varphi)x\big),$$

which is equivalent to

$$x - \mathcal{K}_n^M(x) = \varphi - \mathcal{K}(\varphi) = f.$$

Hence, the modified collocation solution φ_n^M is a fixed point of B_n, that is,

$$B_n(\varphi_n^M) = \varphi_n^M.$$

Hence,

$$\begin{aligned}
\varphi_n^M - \varphi &= B_n(\varphi_n^M) - \varphi \\
&= -(\mathcal{I}-\mathcal{K}'(\varphi))^{-1}\Big(\mathcal{K}'(\varphi) - \mathcal{K}'(\varphi)\varphi - \mathcal{K}_n^M(\varphi_n^M) + \mathcal{K}'(\varphi)\varphi_n^M\Big).
\end{aligned}$$

Multiplying by $\mathcal{K}'(\varphi)$ and using the commutativity of $\mathcal{K}'(\varphi)$ with $(\mathcal{I}-\mathcal{K}'(\varphi))^{-1}$, we obtain

$$\begin{aligned}
\mathcal{K}'(\varphi)(\varphi_n^M-\varphi) = &-(\mathcal{I}-\mathcal{K}'(\varphi))^{-1}\mathcal{K}'(\varphi)\big(\mathcal{K}'(\varphi) - \mathcal{K}_n^M(\varphi)\big) \\
&+ (\mathcal{I}-\mathcal{K}'(\varphi))^{-1}\mathcal{K}'(\varphi)\big(\mathcal{K}_n^M(\varphi_n^M) - \mathcal{K}_n^M(\varphi) - (\mathcal{K}_n^M)'(\varphi)(\varphi_n^M-\varphi)\big) \\
&+ (\mathcal{I}-\mathcal{K}'(\varphi))^{-1}\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)(\varphi_n^M-\varphi).
\end{aligned}$$

By Lemma 3.2.5, the first term is of order $O(h^{4r+4})$. By Lemma 3.2.6, the third term is of order $O(h^{6r+4})$. The second term can be estimated by applying Lemma 3.3 of Grammont et al. [33], which yields

$$\left\|\mathcal{K}_n^M(\varphi_n^M) - \mathcal{K}_n^M(\varphi) - (\mathcal{K}_n^M)'(\varphi)(\varphi_n^M-\varphi)\right\|_\infty = O\Big(\left\|\varphi_n^M-\varphi\right\|_\infty^2\Big) = O\big(h^{8r+6}\big)\,.$$

Hence,

$$\left\|\mathcal{K}'(\varphi)(\varphi_n^M - \varphi)\right\|_\infty = O\big(h^{4r+4}\big)\,.$$

Finally, combining this estimate with (3.2.16), we conclude that

$$\left\|\varphi_n^M - \varphi\right\|_\infty = O\big(h^{4r+4}\big)\,,$$

which completes the proof. □

Theorem 25 shows that the iterated modified collocation solution converges with order $O(h^{4r+4})$. This order is higher than that of the modified collocation solution and is achieved without refining the projection space or increasing the polynomial degree. The analysis demonstrates that the iterated modified collocation method exhibits a superior convergence rate compared to the modified collocation approach. Notably, this improvement is obtained without requiring Gauss collocation points and yields a higher order than all collocation methods considered herein (i.e. collocation, iterated collocation and modified collocation methods).

3.3 Convergence Analysis for Green's Kernels

Let $\mathcal{X} = \mathcal{L}^\infty[0,1]$. In this section, we study the convergence analysis of the Urysohn integral equation with Green's function type kernels defined by (3.1.2). Specifically, the associated Urysohn integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{C}[0,1]$ is given by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1],\ x \in \mathcal{X},$$

where the kernel $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ is a real-valued continuous function.

Although, for an integer $r \geq 0$, whenever $\mathcal{X}_n$ denote the space of piecewise polynomials of degree $\leq 2r$ associated with a uniform partition of $[0,1]$ with mesh length $h = \dfrac{1}{n}$. The interpolatory projection $\mathcal{Q}_n$ defined at $2r+1$ Gauss points on each subinterval $\Delta_j = [t_{j-1}, t_j]$. Under suitable smoothness assumptions on the kernel κ of the Urysohn integral operator, Atkinson-Potra [10] showed that, for $\alpha \geq 2r+1$, the collocation and iterated collocation solutions satisfy

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,, \qquad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{\min(\alpha,\,4r+2,\,2r+3)}\big)\,.$$

With additional regularity assumptions on higher-order derivatives of κ, Kulkarni-Nidhin [57] established that the modified collocation method satisfies

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{2r+3}\big)\,,$$

and that the iterated modified collocation method achieves

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = \begin{cases} O(h^4)\,, & r = 0, \\ O(h^{4r+3})\,, & r \geq 1 \text{ and } \alpha \geq 2r+3. \end{cases}$$

These results indicate that, while the standard and iterated collocation methods exhibit limited improvement, the iterated modified collocation method attains the highest order of convergence among the considered schemes.

In this section, we employ an approximation space $\mathcal{X}_n$ consisting of piecewise polynomials of even degree not exceeding $2r$, together with equidistant collocation points. Recall that $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denotes the interpolatory projection defined at $(2r+1)n$ equally spaced points in $[0,1]$, as introduced in (3.1.3). A key feature of the proposed approach is that it achieves higher-order convergence while relying solely on even-degree piecewise polynomial spaces and equidistant collocation nodes, which are not required to coincide with the zeros of special functions. The enhanced convergence rates are obtained under appropriate smoothness assumptions on the kernel of the underlying integral operator.

Let $\alpha \geq 1$ be an integer. Throughout this section, we assume that the kernel κ associated with the operator $\mathcal{K}$ defined in (3.1.1) satisfies the following properties:

1. For $i = 1, 2, 3, 4$, the functions κ, $\dfrac{\partial^i \kappa}{\partial u^i} \in \mathcal{C}(\Omega \times \mathbb{R})$.
2. Divide Ω into two parts as $\Omega_1 \times \mathbb{R} = \{(s,t,u) : 0 \leq t \leq s \leq 1, u \in \mathbb{R}\}$ and $\Omega_2 \times \mathbb{R} = \{(s,t,u) : 0 \leq s \leq t \leq 1, u \in \mathbb{R}\}$. There are two functions $\kappa_i \in \mathcal{C}^\alpha(\Omega_i \times \mathbb{R})$, $i = 1, 2$, such that
$$\kappa(s,t,u) = \begin{cases} \kappa_1(s,t,u) & \text{if } (s,t,u) \in \Omega_1 \times \mathbb{R}, \\ \kappa_2(s,t,u) & \text{if } (s,t,u) \in \Omega_2 \times \mathbb{R}. \end{cases}$$
3. Let
$$\ell(s,t,u) = \frac{\partial \kappa}{\partial u}(s,t,u), \quad m(s,t,u) = \frac{\partial^2 \kappa}{\partial u^2}(s,t,u), \quad \text{for all } (s,t,u) \in \Omega \times \mathbb{R}.$$
Clearly, ℓ, $m \in \mathcal{C}(\Omega \times \mathbb{R})$. The partial derivatives of $\ell(s,t,u)$ and $m(s,t,u)$ with respect to s and t exhibit jump discontinuities along the line $s = t$.
4. There exist functions ℓ_i, $m_i \in \mathcal{C}^{2r+1}(\Omega_i \times \mathbb{R})$, $i = 1, 2$, such that
$$\ell(s,t,u) = \begin{cases} \ell_1(s,t,u) & \text{if } (s,t,u) \in \Omega_1 \times \mathbb{R}, \\ \ell_2(s,t,u) & \text{if } (s,t,u) \in \Omega_2 \times \mathbb{R} \end{cases}$$

and

$$m(s,t,u) = \begin{cases} m_1(s,t,u) & \text{if } (s,t,u) \in \Omega_1 \times \mathbb{R}, \\ m_2(s,t,u) & \text{if } (s,t,u) \in \Omega_2 \times \mathbb{R}. \end{cases}$$

Since $\dfrac{\partial^i \kappa}{\partial u^i} \in \mathcal{C}(\Omega \times \mathbb{R})$ for $i = 1,2,3,4$, the operator $\mathcal{K}$ being four times Fréchet differentiable at $x \in \mathcal{X}$ is expressed as

$$\mathcal{K}^{(i)}(x)(y_1,\ldots,y_i)(s) = \int_0^1 \frac{\partial^i \kappa}{\partial u^i}(s,t,x(t))y_1(t)\cdots y_i(t)\,dt, \quad s \in [0,1],$$

where

$$\frac{\partial^i \kappa}{\partial u^i}(s,t,x(t)) = \frac{\partial^i \kappa}{\partial u^i}(s,t,u)\bigg|_{u=x(t)}, \quad i = 1,2,3,4,$$

and $y_1, y_2, y_3, y_4 \in \mathcal{X}$. Note that $\mathcal{K}'(x) : \mathcal{X} \to \mathcal{X}$ is a compact linear operator, while $\mathcal{K}^{(i)}(x) : \mathcal{X}^i \to \mathcal{X}$ are multilinear operators, where $\mathcal{X}^i$ represents the Cartesian product of i copies of $\mathcal{X}$ (see Rall [73] or Chapter 1). The norms of these operators are given by

$$\left\|\mathcal{K}^{(i)}(x)\right\| = \sup\left\{\left\|\mathcal{K}^{(i)}(x)(y_1,\ldots,y_i)\right\|_\infty : \|y_j\|_\infty \le 1,\ j = 1,\ldots,i\right\}$$

for $i = 1,2,3,4$. It follows that

$$\left\|\mathcal{K}^{(i)}(x)\right\| \le \sup_{s,t\in[0,1]} \left|\frac{\partial^i \kappa}{\partial u^i}(s,t,x(t))\right|, \quad i = 1,2,3,4.$$

Under the above stated assumptions on kernel κ, the Fréchet derivative of $\mathcal{K}$ at $y \in \mathcal{X}$ is given by

$$(\mathcal{K}'(\varphi)y)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,\varphi(t))y(t)\,dt = \int_0^1 \ell(s,t,\varphi(t))y(t)\,dt, \quad s \in [0,1].$$

As it is assumed in Section 3.1 that the equation $x - \mathcal{K}(x) = f$, with $f \in \mathcal{X}$, admits a unique solution φ. We seek numerical approximations of this solution. Let $f \in \mathcal{C}^{2r+1}[0,1]$. Then, under the above regularity assumptions on the kernel κ following Atkinson-Potra [10, Corollary 3.2], we have $\varphi \in \mathcal{C}^{2r+1}[0,1]$.

For $\delta_0 > 0$, recall that $\mathcal{B}(\varphi,\delta_0) = \{x \in \mathcal{X} : \|x-\varphi\|_\infty < \delta_0\}$ be a closed ball with centre φ and radius δ_0. If $x, y \in \mathcal{B}(\varphi,\delta_0)$, then it can be easily shown that

$$\|\mathcal{K}'(x) - \mathcal{K}'(y)\| \le \gamma \|x-y\|_\infty, \tag{3.3.1}$$

for some constant γ.

We use the following notations throughout the section:

Define $S_1 = \{(s,t) : 0 \leq t \leq s \leq 1\}$, $S_2 = \{(s,t) : 0 \leq s \leq t \leq 1\}$ and

$$\ell_*(s,t) := \ell(s,t,\varphi(t)) = \begin{cases} \ell_{1,*}(s,t) = \ell_1(s,t,\varphi(t)), & (s,t) \in S_1, \\ \ell_{2,*}(s,t) = \ell_2(s,t,\varphi(t)), & (s,t) \in S_2. \end{cases}$$

By assumption $\ell_* \in \mathcal{C}(\Omega)$. Since $\varphi \in \mathcal{C}^{2r+1}[0,1]$, it follows that $\ell_{1,*} \in \mathcal{C}^{2r+1}(S_1)$ and $\ell_{2,*} \in \mathcal{C}^{2r+1}(S_2)$.

For j,k non-negative integers, we set

$$D^{(j,k)}\ell_{i,*}(s,t) = \frac{\partial^{j+k}\ell_{i,*}}{\partial s^j \partial t^k}(s,t), \quad \text{for } i = 1,2.$$

Define

$$C_7 = \max_{0\leq i\leq 4}\left(\sup_{\substack{s,t\in[0,1]\\ |u|\leq\|\varphi\|_\infty+\delta_0}}\left|\frac{\partial^i \kappa}{\partial u^i}(s,t,u)\right|\right),$$

$$C_8 = \max_{\substack{1\leq k\leq n\\ j=1,2}}\left\{\sup_{t_{k-1}\leq t\leq s\leq t_k}\left|D^{(j,0)}\ell_{1,*}(s,t)\right|, \sup_{t_{k-1}\leq s\leq t\leq t_k}\left|D^{(j,0)}\ell_{2,*}(s,t)\right|\right\}.$$

From the Fréchet derivatives of $\mathcal{K}$, it is easy to see that

$$\left\|(\mathcal{K}'(\varphi)y)^{(i)}\right\|_\infty \leq C_7\|y\|_\infty,\ i = 0,1,2, \tag{3.3.2}$$

$$\left\|(\mathcal{K}''(\varphi)(y_1,y_2))^{(i)}\right\|_\infty \leq C_7\|y_1\|_\infty\|y_2\|_\infty,\ i = 0,1,2. \tag{3.3.3}$$

The analysis of interpolatory projection methods based on equidistant collocation points, rather than Gaussian points, for Urysohn integral operators with Green's function type kernels appears to be absent from the existing literature. In particular, there is no detailed study of such projections onto spaces of piecewise polynomials of even degree not exceeding $2r$ over a uniform partition of the interval $[0,1]$. The purpose of this section is to address this gap and to develop the analytical tools required for the convergence analysis in this setting.

3.3.1 Divided Difference Bounds

Let $x \in \mathcal{C}[0,1]$, and let $\{\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}\} \subset [0,1]$ be a set of $2r+1$ distinct collocation points. For any $s \in [0,1]$, we denote by $\left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s\right] x$ the divided difference of the function x at the points $\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}$ and s. Recall that the Fréchet derivative of the Urysohn integral operator $\mathcal{K}$ at φ is given by

$$(\mathcal{K}'(\varphi)x)(s) = \int_0^1 \ell_*(s,t)\, x(t)\, dt, \quad s \in [0,1],$$

where ℓ_* denotes the associated Green's function type kernel. A crucial observation is that the divided difference operator with respect to the first variable may be interchanged with the integral defining $\mathcal{K}'(\varphi)$. More precisely, for any $s \in [0,1]$, we have

$$\left[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, s\right] \left(\mathcal{K}'(\varphi)x\right) = \int_0^1 \left[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, s\right] \ell_*(\cdot, t)\, x(t)\, dt,$$

where $\left[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, s\right] \ell_*(\cdot, t)$ denotes the divided difference of the kernel function ℓ_* with respect to its first argument, with t treated as a parameter.

In this subsection, we establish two fundamental lemmas concerning the divided difference $\left[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, s\right] \left(\mathcal{K}'(\varphi)x\right)$, which are essential for the convergence analysis of the interpolatory projection method at equidistant collocation points.

Lemma 3.3.1. *Let $r \geq 0$ and $\left\{\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}\right\}$ be the set of collocation points in $[t_{j-1}, t_j]$ for $j = 1, 2, \dots, n$. Then*

$$\sup_{s \in [t_{j-1}, t_j]} \left| \left[\tau_j^0, \tau_j^1, \dots, \tau_j^{2r}, s\right] (\mathcal{K}'(\varphi)x) \right| \leq C_{11} \|x\|_\infty h^{-2r},$$

for some constant C_{11}.

Proof. We recall that the interpolation points within each subinterval $[t_{j-1}, t_j]$ are defined as

$$\tau_j^i = t_{j-1} + \frac{ih}{2r} = t_{j-1} + h\zeta_i, \quad \text{for } i = 0, 1, \dots, 2r; \quad j = 1, 2, \dots, n,$$

where $\zeta_i = \dfrac{i}{2r} \in [0,1]$, so that $\tau_j^i \in [t_{j-1}, t_j]$. Also, the action of the Fréchet derivative of the integral operator on a function x is given by

$$(\mathcal{K}'(\varphi)x)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s, t, \varphi(t))\, x(t)\, dt = \int_0^1 \ell_*(s,t)\, x(t)\, dt, \quad s \in [0,1],$$

where

$$\ell_*(s,t) \;=\; \begin{cases} \ell_{1,*}(s,t) & 0 \leq t \leq s \leq 1, \\ \ell_{2,*}(s,t) & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $s \in [t_{j-1}, t_j]$, then splitting the integral over all subintervals gives

$$\begin{aligned}[\tau_j^0, \tau_j^1, s](\mathcal{K}'(\varphi)x) &= \int_0^1 [\tau_j^0, \tau_j^1, s]\ell_*(., t)x(t)\ dt \\ &= \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{t_{k-1}}^{t_k} [\tau_j^0, \tau_j^1, s]\ell_*(., t)x(t)\ dt + \int_{t_{j-1}}^{t_j} [\tau_j^0, \tau_j^1, s]\ell_*(., t)x(t)\ dt.\end{aligned} \tag{3.3.4}$$

Since ℓ_* is sufficiently differentiable on the interval $[t_{k-1}, t_k]$ for $k \neq j$, we have

$$\left| \int_{t_{k-1}}^{t_k} [\tau_j^0, \tau_j^1, s]\ell_*(., t)x(t)dt \right| \leq C_8 \|x\|_\infty h. \tag{3.3.5}$$

To estimate the remaining integral over $[t_{j-1}, t_j]$, we bound $[\tau_j^0, \tau_j^1, s]\ell_*(\cdot, t)$ on that interval. We write $[\tau_j^0, s]\ell_*(\cdot, t)$ explicitly and treat three cases. Fix $\tau_j^0, \tau_j^1, s \in [t_{j-1}, t_j]$.

Case (I) Whenever $t_{j-1} \leq s < \tau_j^0$. Note that

$$[\tau_j^0, s]\ell_*(., t) = \frac{\ell_*(s, t) - \ell_*(\tau_j^0, t)}{s - \tau_j^0} = \begin{cases} \dfrac{\ell_{1,*}(s, t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} & \text{if } t_{j-1} \leq t \leq s, \\ \dfrac{\ell_{2,*}(s, t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} & \text{if } s < t < \tau_j^0, \\ \dfrac{\ell_{2,*}(s, t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0} & \text{if } \tau_j^0 \leq t \leq t_j. \end{cases}$$

Thus, for a fixed $s \in [t_{j-1}, \tau_j^0]$, the function $[\tau_j^0, s]\ell_*(\cdot, t)$ is continuous on $[t_{j-1}, t_j]$. Since $\ell_{1,*}(\cdot, t)$ and $\ell_{2,*}(\cdot, t)$ are continuous on $[s, \tau_j^0]$ and differentiable on (s, τ_j^0), we apply the mean value theorem to obtain

$$\frac{\ell_{1,*}(s, t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\ell_{1,*}(\eta_1, t), \quad \text{and} \quad \frac{\ell_{2,*}(s, t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\ell_{2,*}(\eta_2, t);$$

for some $\eta_1, \eta_2 \in (s, \tau_j^0)$.

Now, for $t \in [t_{j-1}, s]$, we estimate

$$\left| \frac{\ell_{1,*}(s, t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} \right| \leq \left(\sup_{t_{j-1} \leq t \leq s \leq t_j} \left| D^{(1,0)}\ell_{1,*}(s, t) \right| \right) = C_8.$$

Similarly, for $t \in [\tau_j^0, t_j]$, we have

$$\left| \frac{\ell_{2,*}(s, t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0} \right| \leq \left(\sup_{t_{j-1} \leq s \leq t \leq t_j} \left| D^{(1,0)}\ell_{2,*}(s, t) \right| \right) = C_8.$$

Hence, the divided difference is bounded as

$$\left|[\tau_j^0, s]\ell_*(\cdot, t)\right| \leq C_8, \quad \text{for } t \in [t_{j-1}, s] \cup [\tau_j^0, t_j]. \tag{3.3.6}$$

Now, for $s < t < \tau_j^0$, we first compute

$$\begin{aligned} \frac{\ell_{2,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} &= \frac{\ell_{2,*}(s,t) - \ell_{2,*}(t,t) + \ell_{1,*}(t,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} \\ &= \frac{D^{(1,0)}\ell_{2,*}(\eta_3, t)(s-t)}{s - \tau_j^0} + \frac{D^{(1,0)}\ell_{1,*}(\eta_4, t)(t - \tau_j^0)}{s - \tau_j^0}, \end{aligned}$$

for some $\eta_3 \in (s,t)$ and $\eta_4 \in (t, \tau_j^0)$. Therefore,

$$\left|\frac{\ell_{2,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0}\right| \leq C_8 + C_8 = 2C_8.$$

Thus,

$$\left|[\tau_j^0, s]\ell_*(\cdot, t)\right| \leq 2C_8, \quad \text{for } s < t < \tau_j^0.$$

Combining this with the bound in (3.3.6), we obtain

$$\sup_{t \in [t_{j-1}, t_j]} \left|[\tau_j^0, s]\ell_*(\cdot, t)\right| \leq 2C_8. \tag{3.3.7}$$

Case (II) When $s = \tau_j^0$, the divided difference simplifies to the partial derivative

$$[\tau_j^0, \tau_j^0]\ell_*(.,t) = \begin{cases} D^{(1,0)}\ell_{1,*}(\tau_j^0, t) & \text{if } t_{j-1} \leq t < \tau_j^0 < 1, \\ D^{(1,0)}\ell_{2,*}(\tau_j^0, t) & \text{if } t_{j-1} \leq \tau_j^0 < t < 1. \end{cases}$$

Therefore, we obtain the following bound

$$\sup_{t \in [t_{j-1}, t_j]} \left|[\tau_j^0, \tau_j^0]\ell_*(\cdot, t)\right| \leq C_8. \tag{3.3.8}$$

Case (III) Whenever $\tau_j^0 < s \leq t_j$, we proceed in the same manner as in Case (I) and obtain

$$\sup_{t \in [t_{j-1}, t_j]} \left|[\tau_j^0, s]\ell_*(.,t)\right| \leq 2C_8.$$

Combining this with the estimates from (3.3.7) and (3.3.8), we conclude

$$\sup_{s,t \in [t_{j-1}, t_j]} \left|[\tau_j^0, s]\ell_*(\cdot, t)\right| \leq 2C_8. \tag{3.3.9}$$

Now observe that

$$[\tau_j^0, \tau_j^1, s]\ell_*(.,t) = \frac{[\tau_j^0, s]\ell_*(.,t) - [\tau_j^1, s]\ell_*(.,t)}{\tau_j^0 - \tau_j^1} = \frac{1}{h(\zeta_0 - \zeta_1)}\Big[[\tau_j^0, s]\ell_*(.,t) - [\tau_j^1, s]\ell_*(.,t)\Big],$$

which implies

$$\left|[\tau_j^0,\tau_j^1,s]\ell_*(.,t)\right| \le \frac{1}{h|\zeta_0-\zeta_1|}\left[\,\left|[\tau_j^0,s]\ell_*(.,t)\right| + \left|[\tau_j^1,s]\ell_*(.,t)\right|\,\right].$$

It follows from (3.3.9) that

$$\sup_{s,t\in[t_{j-1},t_j]}\left|[\tau_j^0,\tau_j^1,s]\ell_*(.,t)\right| \le \frac{4C_8}{h|\zeta_0-\zeta_1|}.$$

Therefore, we estimate the integral as

$$\left|\int_{t_{j-1}}^{t_j}[\tau_j^0,\tau_j^1,s]\ell_*(.,t)x(t)\ dt\right| \le \frac{4C_8}{h|\zeta_0-\zeta_1|}\|x\|_\infty = \frac{C_9}{h}\|x\|_\infty,$$

where $C_9 = \dfrac{4C_8}{|\zeta_0-\zeta_1|}$ ia constant. Combining this with the estimate from (3.3.5) and substituting into (3.3.4), we obtain

$$\left|[\tau_j^0,\tau_j^1,s](\mathcal{K}'(\varphi)x)\right| \le \left(C_8h+\frac{C_9}{h}\right)\|x\|_\infty \le \frac{C_{10}}{h}\|x\|_\infty,$$

for some positive constant C_{10}. Applying mathematical induction, we finally derive the required estimate:

$$\sup_{s\in[t_{j-1},t_j]}\left|[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},s](K'(\varphi)x)\right| \le C_{11}\|x\|_\infty\frac{1}{h^{2r}},$$

for some positive constant C_{11}. Since $s\in[t_{j-1},t_j]$ was arbitrary, this completes the proof. □

Lemma 3.3.2. *Let $r\ge 0$ and $\left\{\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r}\right\}$ be the set of collocation points in $[t_{j-1},t_j]$ for $j=1,2,\ldots,n$. Then*

$$\sup_{s\in[t_{j-1},t_j]}\left|\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},s,s\right]\left(\mathcal{K}'(\varphi)x\right)\right| \le C_{13}\|x\|_\infty h^{-2r},$$

for some constant C_{13}.

Proof. Let $s\in[t_{j-1},t_j]$. Then, we consider repeated nodes

$$\begin{aligned}[\tau_j^0,s,s](K'(\varphi)x) &= \int_0^1[\tau_j^0,s,s]\,\ell_*(\cdot,t)\,x(t)\,dt\\ &= \sum_{\substack{k=1\\k\ne j}}^{n}\int_{t_{k-1}}^{t_k}[\tau_j^0,s,s]\,\ell_*(\cdot,t)\,x(t)\,dt + \int_{t_{j-1}}^{t_j}[\tau_j^0,s,s]\,\ell_*(\cdot,t)\,x(t)\,dt. \quad (3.3.10)\end{aligned}$$

Since ℓ_* is sufficiently differentiable over each subinterval $[t_{k-1}, t_k]$ for $k \neq j$, we obtain the estimate

$$\left| \int_{t_{k-1}}^{t_k} [\tau_j^0, s, s]\ell_*(., t)x(t)dt \right| \leq C_8 \|x\|_\infty h. \tag{3.3.11}$$

To proceed, it remains to bound the quantity $[\tau_j^0, s, s]\ell_*(\cdot, t)$ for $\tau_j^0, s \in [t_{j-1}, t_j]$. Fix $s \in [t_{j-1}, t_j]$.

Case (I) Whenever $t_{j-1} \leq s < \tau_j^0$, note that

$$[\tau_j^0, s, s]\ell_*(., t) = \frac{[s, s]\ell_*(., t) - [\tau_j^0, s]\ell_*(., t)}{s - \tau_j^0}.$$

Thus,

$$[\tau_j^0, s, s]\ell_*(., t) = \begin{cases} \dfrac{\frac{\partial \ell_{1,*}}{\partial s}(s,t) - \left(\frac{\ell_{1,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0}\right)}{s - \tau_j^0} & \text{if } t_{j-1} \leq t \leq s, \\ \dfrac{\frac{\partial \ell_{2,*}}{\partial s}(s,t) - \left(\frac{\ell_{2,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0}\right)}{s - \tau_j^0} & \text{if } s < t < \tau_j^0, \\ \dfrac{\frac{\partial \ell_{2,*}}{\partial s}(s,t) - \left(\frac{\ell_{2,*}(s,t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0}\right)}{s - \tau_j^0} & \text{if } \tau_j^0 \leq t \leq t_j. \end{cases}$$

The function $[\tau_j^0, s, s]\, \ell_*(\cdot, t)$ may be discontinuous at $t = s$. Since $\ell_{1,*}(\cdot, t)$ and $\ell_{2,*}(\cdot, t)$ are continuous on $[s, \tau_j^0]$ and differentiable on (s, τ_j^0), by the mean value theorem we get

$$\frac{\ell_{1,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\ell_{1,*}(\nu_1, t); \quad \text{and} \quad \frac{\ell_{2,*}(s,t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0} = D^{(1,0)}\ell_{2,*}(\nu_2, t),$$

for some $\nu_1, \nu_2 \in (s, \tau_j^0)$.

Now, for $t \in [t_{j-1}, s]$,

$$\begin{aligned} \left| \frac{\frac{\partial \ell_{1,*}}{\partial s}(s,t) - \left(\frac{\ell_{1,*}(s,t) - \ell_{1,*}(\tau_j^0, t)}{s - \tau_j^0}\right)}{s - \tau_j^0} \right| &= \left| \frac{D^{(1,0)}\ell_{1,*}(s,t) - D^{(1,0)}\ell_{1,*}(\nu_1, t)}{s - \tau_j^0} \right| \\ &= \left| \frac{D^{(2,0)}\ell_{1,*}(\nu_3, t)(s - \nu_1)}{s - \tau_j^0} \right| \\ &\leq \left(\sup_{t_{j-1} \leq t \leq s \leq t_j} \left| D^{(2,0)}\ell_{1,*}(s,t) \right| \right) = C_8, \end{aligned}$$

for some $\nu_3 \in (s, \nu_1)$. Similarly, for $t \in [\tau_j^0, t_j]$, one can see that

$$\left| \frac{\frac{\partial \ell_{2,*}}{\partial s}(s,t) - \left(\frac{\ell_{2,*}(s,t) - \ell_{2,*}(\tau_j^0, t)}{s - \tau_j^0}\right)}{s - \tau_j^0} \right| \leq \left(\sup_{t_{j-1} \leq s \leq t \leq t_j} \left| D^{(2,0)}\ell_{2,*}(s,t) \right| \right) = C_8.$$

Hence,

$$\left|[\tau_j^0, s, s]\,\ell_*(\cdot,t)\right| \le C_8, \quad \text{for } t \in [t_{j-1}, s] \cup [\tau_j^0, t_j]. \tag{3.3.12}$$

Now for $s < t < \tau_j^0$,

$$\begin{aligned}\frac{\ell_{2,*}(s,t) - \ell_{1,*}(\tau_j^0,t)}{s-\tau_j^0} &= \frac{\ell_{2,*}(s,t) - \ell_{2,*}(t,t) + \ell_{1,*}(t,t) - \ell_{1,*}(\tau_j^0,t)}{s-\tau_j^0} \\ &= \frac{D^{(1,0)}\ell_{2,*}(\nu_5,t)(s-t)}{s-\tau_j^0} + \frac{D^{(1,0)}\ell_{1,*}(\nu_6,t)(t-\tau_j^0)}{s-\tau_j^0},\end{aligned}$$

ffor some $\nu_5 \in (s,t)$, $\nu_6 \in (t,\tau_j^0)$. Therefore,

$$\begin{aligned}&\left|\frac{\partial \ell_{2,*}}{\partial s}(s,t) - \left(\frac{\ell_{2,*}(s,t) - \ell_{1,*}(\tau_j^0,t)}{s-\tau_j^0}\right)\right| \\ &\qquad = \left|D^{(1,0)}\ell_{2,*}(s,t) - \frac{D^{(1,0)}\ell_{2,*}(\nu_5,t)(s-t)}{s-\tau_j^0} - \frac{D^{(1,0)}\ell_{1,*}(\nu_6,t)(t-\tau_j^0)}{s-\tau_j^0}\right| \le 3C_8.\end{aligned}$$

This yields

$$\int_s^{\tau_j^0} \left|\frac{\frac{\partial \ell_{2,*}}{\partial s}(s,t) - \left(\frac{\ell_{2,*}(s,t)-\ell_{1,*}(\tau_j^0,t)}{s-\tau_j^0}\right)}{s-\tau_j^0}\right| dt \le 3C_8.$$

Combining with (3.3.12), we get

$$\left|\int_{t_{j-1}}^{t_j} [\tau_j^0, s, s]\ell_*(.,t)x(t)\ dt\right| \le (C_8 + 3C_8 + C_8)\,\|x\|_\infty = 5C_8\|x\|_\infty.$$

Hence, using bound of (3.3.11) and the above estimate in (3.3.10), we conclude

$$\left|[\tau_j^0, s, s](\mathcal{K}'(\varphi)x)\right| \le (C_8 h + 5C_8)\,\|x\|_\infty \le 6C_8\|x\|_\infty. \tag{3.3.13}$$

Case (II) Whenever $s = \tau_j^0$, the third-order divided difference of the function $(K'(\varphi)x)(s)$ at the repeated point τ_j^0, denoted by $[\tau_j^0, \tau_j^0, \tau_j^0](K'(\varphi)x)$, is defined as

$$\begin{aligned}[\tau_j^0, \tau_j^0, \tau_j^0](\mathcal{K}'(\varphi)x) &= \frac{1}{2}(\mathcal{K}'(\varphi)x)''(\tau_j^0) \\ &= \frac{1}{2}\left(\frac{\partial \ell_{1,*}}{\partial s}(\tau_j^0,\tau_j^0) - \frac{\partial \ell_{2,*}}{\partial s}(\tau_j^0,\tau_j^0)\right)x(\tau_j^0) \\ &\quad + \frac{1}{2}\left[\int_{t_j-1}^{\tau_j^0} \frac{\partial^2 \ell_{1,*}}{\partial s^2}(\tau_j^0,t)x(t)\ dt + \int_{\tau_j^0}^{t_j} \frac{\partial^2 \ell_{2,*}}{\partial s^2}(\tau_j^0,t)x(t)\ dt\right].\end{aligned}$$

Hence,

$$\left|[\tau_j^0, \tau_j^0, \tau_j^0]\mathcal{K}'(\varphi)x\right| \le 2C_8\|x\|_\infty. \tag{3.3.14}$$

Case (III) Suppose $\tau_j^0 < s \le t_j$. Proceeding similarly as in Case (I), we obtain

$$\left|[\tau_j^0, s, s](\mathcal{K}'(\varphi)x)\right| \le (C_8 h + 5C_8)\, \|x\|_\infty \le 6C_8 \|x\|_\infty.$$

Combining this with inequalities (3.3.13) and (3.3.14), we derive the following bound

$$\sup_{s\in[t_{j-1},t_j]} \left|[\tau_j^0, s, s](\mathcal{K}'(\varphi)x)\right| \le 6C_8 \|x\|_\infty, \tag{3.3.15}$$

which establishes the required estimate for the case $r = 0$.

For $r = 1$, we analyze the higher-order divided difference

$$\begin{aligned}[\tau_j^0, \tau_j^1, s, s](\mathcal{K}'(\varphi)x) &= \frac{[\tau_j^0, s, s](\mathcal{K}'(\varphi)x) - [\tau_j^1, s, s](\mathcal{K}'(\varphi)x)}{\tau_j^0 - \tau_j^1} \\ &= \frac{1}{h(\zeta_0 - \zeta_1)} \Big[[\tau_j^0, s, s](\mathcal{K}'(\varphi)x) - [\tau_j^1, s, s](\mathcal{K}'(\varphi)x) \Big].\end{aligned}$$

This implies the estimate

$$\left|[\tau_j^0, \tau_j^1, s, s](\mathcal{K}'(\varphi)x)\right| \le \frac{1}{h|\zeta_0 - \zeta_1|} \Big[\left|[\tau_j^0, s, s](\mathcal{K}'(\varphi)x)\right| + \left|[\tau_j^1, s, s](\mathcal{K}'(\varphi)x)\right| \Big].$$

Using inequality (3.3.15), we get

$$\sup_{s\in[t_{j-1},t_j]} \left|[\tau_j^0, \tau_j^1, s, s](\mathcal{K}'(\varphi)x)\right| \le \frac{12C_8}{h|\zeta_0 - \zeta_1|} \|x\|_\infty = C_{12} \|x\|_\infty \frac{1}{h},$$

where $C_{12} = \dfrac{12C_8}{|\zeta_0 - \zeta_1|}$ is a constant. By applying mathematical induction, we conclude that

$$\sup_{s\in[t_{j-1},t_j]} \left|[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, s, s](\mathcal{K}'(\varphi)x)\right| \le C_{13} \|x\|_\infty \frac{1}{h^{2r}},$$

for some constant C_{13}, completing the required estimate. □

Further in this section, we will now establish convergence orders for collocation and iteration collocation solutions (i.e. φ_n^C and φ_n^S). Then, we will determine convergence orders for both the modified collocation solution φ_n^M and the iterated modified collocation solution $\widetilde{\varphi}_n^M$.

3.3.2 Collocation and Iterated Collocation Methods

In this subsection, we first establish an analogous result has already been proved in Chapter 2, and the present analysis follows a similar line of reasoning adapted to the current setting and following previous lemma.This result plays a crucial role in the derivation of the main convergence result for the iterated collocation solution.

Lemma 3.3.3. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_{14}\|x\|_{2r+2,\infty}h^{2r+2},$$

where C_{14} is a constant independent of h.

Proof. Let $s \in [0,1]$. Then we have

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) &= \int_0^1 \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_n)x(t)\,dt\\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})x(t)\,dt\\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \ell_*(s,t)\Psi_j(t)\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},t\right]x\,dt,\end{aligned}$$

where

$$\Psi_j(t) = \prod_{i=0}^{2r}(t-\tau_j^i), \quad \text{for } j=1,2,\dots,n.$$

Let $s \in [t_{i-1},t_i] \subset [0,1]$, for some $1 \le i \le n$. Then we write

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) &= \sum_{\substack{j=1\\ j\neq i}}^n \int_{t_{j-1}}^{t_j} \ell_*(s,t)\Psi_j(t)\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},t\right]x\,dt \\ &\quad + \int_{t_{i-1}}^{t_i} \ell_*(s,t)\Psi_i(t)\left[\tau_i^0,\tau_i^1,\dots,\tau_i^{2r},t\right]x\,dt.\end{aligned} \tag{3.3.16}$$

For fix s, let

$$f_j^s(t) = \ell_*(s,t)\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},t\right]x, \quad t\in[t_{j-1},t_j], \text{ for } j=1,2,\dots,n.$$

That is

$$f_j^s(t) = \begin{cases}\ell_{1,*}(s,t)\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},t\right]x, & j \le i,\\ \ell_{2,*}(s,t)\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},t\right]x, & j \ge i.\end{cases}$$

Note that the function f_j^s is continuous on $[t_{j-1},t_j]$ and differentiable on (t_{j-1},t_j) for $j \neq i$. Therefore by Taylor series expansion of f_j^s about the point $s_j = \dfrac{t_{j-1}+t_j}{2}$, there exist a point $\zeta_{j1} \in (t_{j-1},t_j)$ such that

$$(f_j^s)(t) = f_j^s(s_j) + (t-s_j)(f_j^s)'(\zeta_{j1}),$$

where

$$(f_j^s)'(\zeta_{j1}) = \begin{cases}\ell_{1,*}(s,\zeta_{j1})\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},\zeta_{j1},\zeta_{j1}\right]x + D^{(0,1)}\ell_{1,*}(s,\zeta_{j1})\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},\zeta_{j1}\right]x, & j \le i,\\ \ell_{2,*}(s,\zeta_{j1})\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},\zeta_{j1},\zeta_{j1}\right]x + D^{(0,1)}\ell_{2,*}(s,\zeta_{j1})\left[\tau_j^0,\tau_j^1,\dots,\tau_j^{2r},\zeta_{j1}\right]x, & j \ge i.\end{cases}$$

Let

$$C_{14} = \max_{k=0,1,2} \left\{ \sup_{0\le t\le s\le 1} \left| D^{(0,k)} \ell_{1,*}(s,t) \right|, \sup_{0\le s\le t\le 1} \left| D^{(0,k)} \ell_{2,*}(s,t) \right| \right\}.$$

Then

$$\left| (f_j^s)'(\zeta_{j1}) \right| \le 2C_{14} \|x\|_{2r+2,\infty}.$$

Now, we write

$$\int_{t_{j-1}}^{t_j} \ell_*(s,t)\Psi_j(t) \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t\right] x \; dt = \int_{t_{j-1}}^{t_j} (t - s_j) \, (f_j^s)'(\zeta_{j1})\Psi_j(t) \; dt.$$

It follows that

$$\left| \int_{t_{j-1}}^{t_j} \ell_*(s,t)\Psi_j(t) \left[\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t\right] x \; dt \right| \le 2C_{14} \|x\|_{2r+2,\infty} h^{2r+3}.$$

Since $\int_{t_{i-1}}^{t_i} \Psi_i(t) \; dt = 0$,

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \ell_*(s,t)\Psi_i(t) \left[\tau_i^0, \tau_i^1, \ldots, \tau_i^{2r}, t\right] x \; dt &= \int_{t_{i-1}}^{t_i} (f_i^s)(t)\Psi_i(t) \; dt, \;\; s \in [t_{i-1}, t_i] \\ &= \int_{t_{i-1}}^{t_i} [(f_i^s)(t) - (f_i^s)(s)]\Psi_i(t) \; dt. \end{aligned}$$

Since the function f_i^s is continuous on $[t_{i-1}, t_i]$ and differentiable on (t_{i-1}, t_i), by mean value theorem, we have

$$(f_i^s)(t) - (f_i^s)(s) = (t - s)(f_i^s)'(\zeta_{i2}),$$

for some $\zeta_{i2} \in (t_{i-1}, t_i)$. Then

$$\left| \int_{t_{i-1}}^{t_i} \ell_*(s,t)\Psi_i(t) \left[\tau_i^0, \tau_i^1, \ldots, \tau_i^{2r}, t\right] x \; dt \right| \le 2C_{14} \|x\|_{2r+2,\infty} h^{2r+3}.$$

By (3.3.16),

$$\begin{aligned} |\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x(s)| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} &\left| \int_{t_{j-1}}^{t_j} \ell_*(s,t) \left([\tau_j^0, \tau_j^1, \ldots, \tau_j^{2r}, t] x \right) \Psi_j(t) \; dt \right| \\ &+ \left| \int_{t_{i-1}}^{t_i} \ell_*(s,t) \left([\tau_i^0, \tau_i^1, \ldots, \tau_i^{2r}, t] x \right) \Psi_i(t) \; dt \right|. \end{aligned}$$

Hence

$$\left| \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)x(s) \right| \le 2C_{14} \|x\|_{2r+2,\infty} h^{2r+2}.$$

Taking the supremum over $s \in [0,1]$ completes the proof. □

The estimate established above is now employed to determine the convergence orders of the iterated collocation methods for Urysohn integral equations with Green's function type kernels. In particular, the convergence order of the collocation solution φ_n^C of (3.2.2) follows by combining the interpolation error bound (3.1.5) with the stability estimate (3.2.6). Thus, we obtain

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big)\,.$$

Likewise, the convergence order of the iterated collocation solution φ_n^S defined by (3.2.3) is obtained by invoking Lemma 3.3.3 together with the estimate (3.2.7). Hence, we arrive at the following error bound

$$\left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+2}\big)\,.$$

These results clearly indicate that, in the case of Green's function type kernels, the iterated collocation method attains a higher order of convergence than the standard collocation method similar to case of smooth kernels, even though the interpolation nodes are equidistant and not chosen to be Gauss points.

3.3.3 Modified Collocation Method

For $\delta_0 > 0$, let us recall that $\mathcal{B}(\varphi, \delta) = \{x \in \mathcal{X} : \|\varphi - x\|_\infty < \delta_0\}$. As shown earlier, the modified collocation solution φ_n^M exists in a neighborhood of φ.

We begin by establishing the following key result, which plays a crucial role in the proof of Theorem 26, one of the main theorem of this section, where the order of convergence of the modified collocation solution is derived. Henceforth, we assume that n_0 is a positive integer such that, for all $n \geq n_0$, it holds that $\mathcal{Q}_n\varphi \in \mathcal{B}(\varphi, \delta_0)$.

Lemma 3.3.4. *Let $r \geq 1$ and $f \in \mathcal{C}^{2r+1}[0,1]$. Then*

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\big]\right\|_\infty = O\big(h^{4r+3}\big)\,.$$

Proof. Consider the generalised Taylor's theorem upto second-order

$$\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi) = \frac{1}{2}\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi - \varphi)^2 + R(\mathcal{Q}_n\varphi - \varphi),$$

with the remainder term

$$R(\mathcal{Q}_n\varphi - \varphi)(s) = \int_0^1 \left[\mathcal{K}^{(3)}(\varphi + \theta(\mathcal{Q}_n\varphi - \varphi))(\mathcal{Q}_n\varphi - \varphi)^3\right](s)\frac{(1-\theta)^2}{2}\,d\theta, \quad s \in [0,1]. \tag{3.3.17}$$

Recall that $\dfrac{\partial^3 \kappa}{\partial u^3}$ is assumed to be continuous on $\Omega \times \mathbb{R}$, it follows that the operator $\mathcal{K}$ is three times Fréchet differentiable. The third Fréchet derivative of $\mathcal{K}$ is then given by

$$\left[\mathcal{K}^{(3)}(\varphi + \theta(\mathcal{Q}_n\varphi - \varphi))(\mathcal{Q}_n\varphi - \varphi)^3\right](s) = \int_0^1 \frac{\partial^3 \kappa}{\partial u^3}(s,t,\varphi(t) + \theta(\mathcal{Q}_n\varphi - \varphi)(t))(\mathcal{Q}_n\varphi - \varphi)^3(t)\, dt.$$

Then

$$\left|\left[\mathcal{K}^{(3)}(\varphi + \theta(\mathcal{Q}_n\varphi - \varphi))(\mathcal{Q}_n\varphi - \varphi)^3\right](s)\right| \leq \left(\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\delta_0}} \left|\frac{\partial^3 \kappa}{\partial u^3}(s,t,u)\right|\right)\|\mathcal{Q}_n\varphi - \varphi\|_\infty^3.$$

Thus, we obtain

$$\left\|\left[\mathcal{K}^{(3)}(\varphi + \theta(\mathcal{Q}_n\varphi - \varphi))(\mathcal{Q}_n\varphi - \varphi)^3\right]\right\|_\infty \leq C_7\|\mathcal{Q}_n\varphi - \varphi\|_\infty^3.$$

As a consequence, equation (3.3.17) gives

$$\|R(\mathcal{Q}_n\varphi - \varphi)\|_\infty \leq \frac{C_7}{6}\|\mathcal{Q}_n\varphi - \varphi\|_\infty^3. \tag{3.3.18}$$

Note that

$$\begin{aligned}(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)] &= \frac{1}{2}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi - \varphi)^2\\ &\quad + (\mathcal{I} - \mathcal{Q}_n)R(\mathcal{Q}_n\varphi - \varphi).\end{aligned} \tag{3.3.19}$$

Since $f \in \mathcal{C}^{2r+1}[0,1]$, it follows that $\varphi \in \mathcal{C}^{2r+1}[0,1]$. Thus

$$\|\mathcal{Q}_n\varphi - \varphi\|_\infty = O\big(h^{2r+1}\big).$$

As $\mathcal{Q}_n\varphi - \varphi \in \mathcal{C}[0,1]$, we have $\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)^2 \in \mathcal{C}^2[0,1]$, and hence

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi - \varphi)^2\right\|_\infty \leq C_1\left\|\left(\mathcal{K}''(\varphi)\left(\mathcal{Q}_n\varphi - \varphi\right)^2\right)'\right\|_\infty h.$$

By inequality (3.3.3), it holds that

$$\left\|\left(\mathcal{K}''(\varphi)\left(\mathcal{Q}_n\varphi - \varphi\right)^2\right)'\right\|_\infty \leq C_7\|\mathcal{Q}_n\varphi - \varphi\|_\infty^2,$$

which leads to

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi - \varphi)^2\right\|_\infty = O\big(h^{4r+3}\big).$$

Moreover, using the uniform boundedness of the operator norms $\|\mathcal{Q}_n\|$ and estimate (3.3.18), we get

$$\|(\mathcal{I} - \mathcal{Q}_n)R(\mathcal{Q}_n\varphi - \varphi)\|_\infty = O\big(h^{6r+3}\big).$$

Substituting the above bounds into (3.3.19), we obtain the required estimate

$$\left\|(\mathcal{I} - \mathcal{Q}_n)\left[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)\right]\right\|_\infty = O\big(h^{4r+3}\big).$$

$\square$

Theorem 26. *Let $\mathcal{K}$ be a Urysohn integral operator defined as* (3.1.1) *with Green's function type kernel and $f \in \mathcal{C}^{2r+2}[0,1]$ be a given function. Further, assume that* 1 *is not an eigenvalue of $\mathcal{K}'(\varphi)$, and consider the equation* (3.1.2) *with φ as the unique solution in $\mathcal{B}(\varphi, \delta_0)$. Let $\mathcal{Q}_n$ be the interpolatory operator onto the approximating space $\mathcal{X}_n$ defined by* (3.1.3)*. Let $\varphi_n^M \in \mathcal{B}(\varphi, \delta_0)$ be the modified collocation solution of* (3.2.4)*. Then*

$$\left\|\varphi - \varphi_n^M\right\|_\infty = \begin{cases} O(h^3)\,, & r = 0, \\ O(h^{2r+2})\,, & r \geq 1. \end{cases}$$

Proof. Case I: When $r = 0$. See [57].
Case II: When $r \geq 1$.
We write the following estimate from (3.2.10) as

$$\left\|\varphi - \varphi_n^M\right\|_\infty \leq 4\left\|(I - \mathcal{K}'(\varphi))^{-1}\right\| \|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi))\|_\infty. \tag{3.3.20}$$

Further consider

$$\begin{aligned}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi)) = &-(\mathcal{I} - \mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{K}(\varphi) - \mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi - \varphi)] \\ &- (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi. \end{aligned} \tag{3.3.21}$$

Now

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty \leq \|\mathcal{I} - \mathcal{Q}_n\| \|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty.$$

Uniformly boundedness of the sequence $\{\mathcal{Q}_n\}$ and Lemma 3.3.3 implies

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\varphi\|_\infty = O\big(h^{2r+2}\big)\,.$$

Applying above estimate and Lemma 3.3.4 in equation (3.3.21), we obtain

$$\|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}(\varphi) - \mathcal{K}(\mathcal{Q}_n\varphi))\|_\infty = O\big(h^{2r+2}\big)\,.$$

Using the above estimate in (3.3.20), we obtained the desired result as

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{2r+2}\big)\,.$$

This proves the theorem. □

3.3.4 Iterated Modified Collocation Method

In this subsection, we establish the main result for the non-smooth(Green's function type) kernel case. Specifically, we prove that the iterated modified collocation method achieves a higher order of convergence than the modified collocation method. We proceed to prove a series of intermediate results essential for the proof of our main theorem.

Lemma 3.3.5. *If $x \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \leq C_{16}\|x\|_{2r+1,\infty}h^{2r+3},$$

where C_{16} is a constant independent of h.

Proof. For any $s \in [0,1]$,

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) &= \int_0^1 \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(t)\ dt\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(t)\ dt\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\ dt, \qquad (3.3.22)\end{aligned}$$

where

$$g_j^s(t) = \ell_*(s,t)\left(\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right),\ t\in[t_{j-1},t_j],\ \text{for } j=1,2,\ldots,n.$$

Let $s \in [t_{i-1},t_i] \subset [0,1]$, for some $1 \leq i \leq n$. Note that

$$g_j^s(t) \ = \ \begin{cases}\ell_{1,*}(s,t)\left(\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right), & j\leq i,\\ \ell_{2,*}(s,t)\left(\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right), & j\geq i.\end{cases}$$

Denote

$$\left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t] = \left(\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t\right]\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right),$$

and

$$\left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] = \left(\left[\tau_j^0,\tau_j^1,\ldots,\tau_j^{2r},t,t\right]\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right).$$

Note that the function g_j^s is continuous on $[t_{j-1},t_j]$ and differentiable on (t_{j-1},t_j) for $j\neq i$. Then it follows that

$$(g_j^s)'(t) \ = \ \begin{cases}\ell_{1,*}(s,t)\left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] + \left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t]\ D^{(0,1)}\ell_{1,*}(s,t), & j\leq i,\\ \ell_{2,*}(s,t)\left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t,t] + \left(\delta_j^{2r}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x\right)[t]\ D^{(0,1)}\ell_{2,*}(s,t), & j\geq i.\end{cases}$$

Therefore, by Lemma 3.3.1, Lemma 3.3.2 and estimate (3.1.5), we have

$$\left|(g_j^s)'(t)\right| \leq C_{14}C_{15}\|(\mathcal{I}-\mathcal{Q}_n)x\|_\infty h^{-2r} \leq C_1C_{14}C_{15}\left\|x^{(2r+1)}\right\|_\infty h,$$

where $C_{15} > 0$ is a constant independent of h. From (3.3.22), we write

$$\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s) = \sum_{\substack{j=1\\ j\neq i}}^{n} \int_{t_{j-1}}^{t_j} g_j^s(t)\Psi_j(t)\ dt + \int_{t_{i-1}}^{t_i} g_i^s(t)\Psi_i(t)\ dt$$

By employing similar argument as presented in the proof of Lemma 3.3.3, we acquire

$$|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)x(s)| \leq 2C_1C_{14}C_{15}\left\|x^{(2r+1)}\right\|_\infty h^{2r+3}.$$

This proves the lemma. □

Lemma 3.3.6. *If* $x \in \mathcal{C}[0,1]$*, then*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x\|_\infty \leq C_{17}\|x\|_\infty h^2,$$

where C_{17} *is a constant independent of* h*. Further,*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)\| = O\big(h^2\big)\,.$$

Proof. Let $s \in [0,1]$. Then

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x(s) &= \int_0^1 \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x(t)\ dt\\ &= \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \ell_*(s,t)(\mathcal{I}-\mathcal{Q}_{n,j})\mathcal{K}'(\varphi)x(t)\ dt.\end{aligned}$$

Since $x \in \mathcal{C}[0,1]$, then $\mathcal{K}'(\varphi)x \in \mathcal{C}^2[0,1]$, and therefore

$$\begin{aligned}|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x(s)| &\leq \|\ell_*\|_\infty \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \|(\mathcal{I}-\mathcal{Q}_{n,j})\mathcal{K}'(\varphi)x\|_\infty\ dt\\ &\leq \|\ell_*\|_\infty \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} C_1\left\|(\mathcal{K}'(\varphi)x)^{(2)}\right\|_\infty h^2\ dt.\end{aligned}$$

From the estimate (3.3.2), we have

$$|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x(s)| \leq \|\ell_*\|_\infty \sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} C_7C_1\|x\|_\infty h^2\ dt = C_7C_1\|\ell_*\|_\infty\|x\|_\infty h^2.$$

This proves the required result, where $C_{17} = C_7C_1\|\ell_*\|_\infty < \infty$. Thus,

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)\| = \sup_{\|x\|_\infty\leq 1}\left\{\sup_{s\in[0,1]}|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)x(s)|\right\} = O\big(h^2\big)\,,$$

which completes the proof. □

Remark 3. *If $x \in \mathcal{C}[0,1]$, then it can be shown that*

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}''(\varphi)\| = O\big(h^2\big),$$

and

$$\big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}^{(3)}(\varphi)\big\| = O\big(h^2\big).$$

Lemma 3.3.7. *Let $r \geq 1$ and $f \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\big]\big\|_\infty = O\big(h^{4r+4}\big).$$

Proof. Let n_0 be a positive integer such that $n \geq n_0$ implies that $\mathcal{Q}_n\varphi \in \mathcal{B}(\varphi,\delta_0)$. Then by generalized Taylor's theorem, we have

$$\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi) = \frac{1}{2}\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi-\varphi)^2+\frac{1}{3!}\mathcal{K}^{(3)}(\varphi)(\mathcal{Q}_n\varphi-\varphi)^3+R(\mathcal{Q}_n\varphi-\varphi),$$

where the remainder term is given by

$$R(\mathcal{Q}_n\varphi-\varphi)(s) = \int_0^1 \Big[\mathcal{K}^{(4)}(\varphi+\theta(\mathcal{Q}_n\varphi-\varphi))(\mathcal{Q}_n\varphi-\varphi)^4\Big](s)\frac{(1-\theta)^3}{6}\,d\theta,\ \ s\in[0,1]. \quad (3.3.23)$$

Since $\dfrac{\partial^4\kappa}{\partial u^4}$ is continuous on $\Omega\times\mathbb{R}$, the operator $\mathcal{K}$ is four times Fréchet differentiable, and its fourth derivative is given by

$$\Big[\mathcal{K}^{(4)}(\varphi+\theta(\mathcal{Q}_n\varphi-\varphi))(\mathcal{Q}_n\varphi-\varphi)^4\Big](s) = \int_0^1 \frac{\partial^4\kappa}{\partial u^4}\left(s,t,\varphi(t)+\theta(\mathcal{Q}_n\varphi-\varphi)(t)\right)(\mathcal{Q}_n\varphi-\varphi)^4(t)\,dt.$$

Hence, we obtain the uniform bound

$$\left|\Big[\mathcal{K}^{(4)}(\varphi+\theta(\mathcal{Q}_n\varphi-\varphi))(\mathcal{Q}_n\varphi-\varphi)^4\Big](s)\right| \leq \left(\sup_{\substack{s,t\in[0,1]\\|u|\leq\|\varphi\|_\infty+\delta_0}}\left|\frac{\partial^4\kappa}{\partial u^4}(s,t,u)\right|\right)\|\mathcal{Q}_n\varphi-\varphi\|_\infty^4.$$

Thus,

$$\left\|\Big[\mathcal{K}^{(4)}(\varphi+\theta(\mathcal{Q}_n\varphi-\varphi))(\mathcal{Q}_n\varphi-\varphi)^4\Big]\right\|_\infty \leq C_7\|\mathcal{Q}_n\varphi-\varphi\|_\infty^4.$$

Now, from equation (3.3.23), we obtain

$$\|R(\mathcal{Q}_n\varphi-\varphi)\|_\infty \leq \frac{C_7}{24}\|\mathcal{Q}_n\varphi-\varphi\|_\infty^4.$$

Therefore, we can express

$$\begin{aligned}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)] &= \frac{1}{2}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi-\varphi)^2\\ &\quad+\frac{1}{6}\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}^{(3)}(\varphi)(\mathcal{Q}_n\varphi-\varphi)^3\\ &\quad+\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)R(\mathcal{Q}_n\varphi-\varphi). \quad (3.3.24)\end{aligned}$$

Since $f \in \mathcal{C}^{2r+1}[0,1]$, it follows that $\varphi \in \mathcal{C}^{2r+1}[0,1]$. Therefore, $\|Q_n\varphi - \varphi\|_\infty = O\big(h^{2r+1}\big)$, and by Remark 3, we have the following operator norm estimates

$$\big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}''(\varphi)(\mathcal{Q}_n\varphi-\varphi)^2\big\|_\infty \le \big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}''(\varphi)\big\|\|\mathcal{Q}_n\varphi-\varphi\|_\infty^2 = O\big(h^{4r+4}\big)\,, \tag{3.3.25}$$

$$\Big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}^{(3)}(\varphi)(\mathcal{Q}_n\varphi-\varphi)^3\Big\|_\infty \le \Big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}^{(3)}(\varphi)\Big\|\|\mathcal{Q}_n\varphi-\varphi\|_\infty^3 = O\big(h^{6r+5}\big)\,. \tag{3.3.26}$$

Furthermore, we estimate the contribution from the remainder term as

$$\begin{aligned}\big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)R(\mathcal{Q}_n\varphi-\varphi)\big\|_\infty &\le \big\|\mathcal{K}'(\varphi)\big\|\|\mathcal{I}-\mathcal{Q}_n\|\|R(\mathcal{Q}_n\varphi-\varphi)\|_\infty \\ &\le C_{18}\|R(\mathcal{Q}_n\varphi-\varphi)\|_\infty = O\big(h^{8r+4}\big)\,,\end{aligned}$$

where $C_{18} = \|\mathcal{K}'(\varphi)\|\|\mathcal{I}-\mathcal{Q}_n\| < \infty$. Combining estimates (3.3.25), (3.3.26), and the above bound on the remainder term into equation (3.3.24), we conclude that

$$\big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\big]\big\|_\infty = O\big(h^{4r+4}\big)\,.$$

This completes the proof. □

Lemma 3.3.8. *Let $r \ge 1$ and $f \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\big\|\mathcal{K}'(\varphi)\big[\mathcal{K}(\varphi)-\mathcal{K}_n^M(\varphi)\big]\big\|_\infty = O\big(h^{2r+3}\big)\,.$$

Proof. Since the modified operator is given by

$$\mathcal{K}_n^M(\varphi) = \mathcal{Q}_n\mathcal{K}(\varphi) + \mathcal{K}(\mathcal{Q}_n\varphi) - \mathcal{Q}_n\mathcal{K}(\mathcal{Q}_n\varphi),$$

we consider the difference

$$\begin{aligned}\mathcal{K}(\varphi)-\mathcal{K}_n^M(\varphi) &= (\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}(\varphi)-\mathcal{K}(\mathcal{Q}_n\varphi)) \\ &= -(\mathcal{I}-\mathcal{Q}_n)[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)] \\ &\quad + (\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\varphi.\end{aligned}$$

Applying the linear operator $\mathcal{K}'(\varphi)$ to both sides gives

$$\begin{aligned}\big\|\mathcal{K}'(\varphi)\big[\mathcal{K}(\varphi)-\mathcal{K}_n^M(\varphi)\big]\big\|_\infty &\le \big\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\big[\mathcal{K}(\mathcal{Q}_n\varphi)-\mathcal{K}(\varphi)-\mathcal{K}'(\varphi)(\mathcal{Q}_n\varphi-\varphi)\big]\big\|_\infty \\ &\quad + \|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty.\end{aligned} \tag{3.3.27}$$

Using Lemma 3.3.5, the second term is estimated as

$$\|\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}'(\varphi)(\mathcal{I}-\mathcal{Q}_n)\varphi\|_\infty = O\big(h^{2r+3}\big)\,.$$

Applying this result along with the estimate from Lemma 3.3.7 to (3.3.27), we obtain the desired bound. □

Lemma 3.3.9. *Let $r \geq 1$ and $f \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\left\| \mathcal{K}'(\varphi) \left(\left(\mathcal{K}_n^M \right)' (\varphi) - \mathcal{K}'(\varphi) \right) \left(\varphi - \varphi_n^M \right) \right\|_\infty = O\big(h^{2r+4}\big).$$

Proof. We begin by expressing the difference between the derivatives of the modified and original operators as

$$\mathcal{K}'(\varphi) \left((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi) \right) = -\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi) + \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n. \quad (3.3.28)$$

We now decompose the second term on the right-hand side as

$$\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n = \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi))\mathcal{Q}_n + \mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\varphi)\mathcal{Q}_n. \quad (3.3.29)$$

Using the Lipschitz continuity of $\mathcal{K}'$ from (3.3.1), we have

$$\|\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi)\| \leq \gamma \|\mathcal{Q}_n\varphi - \varphi\|_\infty.$$

Since $\|\mathcal{Q}_n\|$ is uniformly bounded and $\|\mathcal{Q}_n\varphi - \varphi\|_\infty = O(h^{2r+1})$, it follows that

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K}'(\mathcal{Q}_n\varphi) - \mathcal{K}'(\varphi))\mathcal{Q}_n\| \leq C_{18}\gamma \|\mathcal{Q}_n\| \|\mathcal{Q}_n\varphi - \varphi\|_\infty = O\left(h^{2r+1}\right).$$

Using the Lemma 3.3.5 and the above estimate in (3.3.29), we obtain

$$\|\mathcal{K}'(\varphi)(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}'(\mathcal{Q}_n\varphi)\mathcal{Q}_n\| = O\big(h^2\big).$$

Substituting back into (3.3.28), and again using Lemma 3.3.5, we conclude

$$\left\|\mathcal{K}'(\varphi) \left((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi) \right)\right\| = O\big(h^2\big).$$

Therefore, we can write

$$\left\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi))(\varphi - \varphi_n^M)\big)\right\|_\infty \leq \left\|\mathcal{K}'(\varphi)\big((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi)\big)\right\| \left\|\varphi_n^M - \varphi\right\|_\infty,$$

and the Theorem 26 give the required result. □

Theorem 27. *Let $\mathcal{K}$ be a Urysohn integral operator defined as* (3.1.1) *with Green's function type kernel and $f \in \mathcal{C}^{2r+1}[0,1]$ be a given function. Further, assume that* 1 *is not an eigenvalue of $\mathcal{K}'(\varphi)$, and consider the equation* (3.1.2) *with φ as the unique solution in $\mathcal{B}(\varphi, \delta_0)$. Let $\mathcal{Q}_n$ be the interpolatory operator onto the approximating space $\mathcal{X}_n$ defined by* (3.1.3)*. Let $\widetilde{\varphi}_n^M \in \mathcal{B}(\varphi, \delta_0)$ be the iterated modified collocation solution defined by* (3.2.5)*. Then*

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = \begin{cases} O(h^4), & r = 0, \\ O(h^{2r+3}), & r \geq 1. \end{cases}.$$

Proof. Case I: When $r = 0$. See [57].
Case II: When $r \geq 1$.
We have $\varphi = \mathcal{K}(\varphi) + f$ and $\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f$, therefore $\widetilde{\varphi}_n^M - \varphi = \mathcal{K}(\varphi_n^M) - \mathcal{K}(\varphi)$. From estimate (3.2.16), we have

$$\left\|\widetilde{\varphi}_n^M - \varphi\right\|_\infty \leq \left\|\mathcal{K}'(\varphi)(\varphi_n^M - \varphi)\right\|_\infty + C_7\left\|\varphi_n^M - \varphi\right\|_\infty^2. \tag{3.3.30}$$

We analyze the term $\mathcal{K}'(\varphi)(\varphi_n^M - \varphi)$ using the identity

$$\begin{aligned}\mathcal{K}'(\varphi)(\varphi_n^M - \varphi) = &- \left(\mathcal{I} - \mathcal{K}'(\varphi)\right)^{-1} \mathcal{K}'(\varphi)\left[\mathcal{K}(\varphi) - \mathcal{K}_n^M(\varphi)\right] \\ &+ \left(\mathcal{I} - \mathcal{K}'(\varphi)\right)^{-1} \mathcal{K}'(\varphi)\left[\mathcal{K}_n^M(\varphi_n^M) - \mathcal{K}_n^M(\varphi) - (\mathcal{K}_n^M)'(\varphi)(\varphi_n^M - \varphi)\right] \\ &+ \left(\mathcal{I} - \mathcal{K}'(\varphi)\right)^{-1} \mathcal{K}'(\varphi)\left[((\mathcal{K}_n^M)'(\varphi) - \mathcal{K}'(\varphi))(\varphi_n^M - \varphi)\right].\end{aligned} \tag{3.3.31}$$

The bound for the second term of the above equation is obtained using Grammont et al. [33, Lemma 3.3] as

$$\left\|\mathcal{K}_n^M(\varphi_n^M) - \mathcal{K}_n^M(\varphi) - (\mathcal{K}_n^M)'(\varphi)(\varphi_n^M - \varphi)\right\|_\infty = O\left(\left\|\varphi_n^M - \varphi\right\|_\infty^2\right) = O\left(h^{4r+4}\right).$$

By Lemmas 3.3.8 and 3.3.9, we find that the first and third terms in equation (3.3.31) are of orders h^{2r+3} and h^{2r+4}, respectively. Therefore,

$$\left\|\mathcal{K}'(\varphi)(\varphi_n^M - \varphi)\right\|_\infty = O\left(h^{2r+3}\right).$$

Substituting into (3.3.30), we obtain the final convergence estimate

$$\left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\left(h^{2r+3}\right),$$

which completes the proof. □

In several approaches, higher-order convergence is often obtained through carefully chosen collocation points, typically associated with special points like Gauss points and that improve numerical integration accuracy. In contrast, the present work adopts a simpler framework based on equidistant collocation points together with even-degree piecewise polynomial spaces. Despite this straightforward construction, the proposed method attains higher-order convergence and exhibits strong theoretical and numerical performance, highlighting its effectiveness and practical appeal.

3.4 Numerical Results

For illustration, we consider two examples: one with a smooth kernel and the other with a non-smooth (Green's function type) kernel. These numerical examples verify the convergence rates obtained in this article. Consider the uniform partition of $[0,1]$ as $0 = t_0 < t_1 < \cdots < t_n = 1$, where $t_j = \dfrac{j}{n}, \quad j = 0, 1, \ldots, n.$

Example 7. *Consider the following nonlinear integral equation*

$$x(s) - \int_0^1 \frac{1}{s + t + x(t)}\, dt = f(s), \quad 0 \le s \le 1,$$

where we choose f such that $\varphi(s) = \dfrac{1}{s+1}$ is the exact solution.

Let $\mathcal{X}_n$ denote the space of piecewise constant functions associated with the uniform partition of $[0,1]$, and let $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ be the interpolatory projection defined at the midpoints, as given by (3.1.3). It is observed that the iterated collocation method achieves a convergence rate of $O(1/n^2)$, while the iterated modified collocation method attains a significantly higher rate of $O(1/n^4)$. We replace all the integrals appeared in the above methods by the composite 2-point Gaussian quadrature rule. In the tables presented, δ_C, δ_S, δ_M, and δ_{IM} denote the computed orders of convergence corresponding to the collocation, iterated collocation, modified collocation, and iterated modified collocation methods, respectively.

Table 3.4.1: Comparison of convergence rates in collocation variants for smooth kernel.

n	$\|\varphi - \varphi_n^C\|_\infty$	δ_C	$\|\varphi - \varphi_n^S\|_\infty$	δ_S	$\|\varphi - \varphi_n^M\|_\infty$	δ_M	$\|\varphi - \widetilde{\varphi}_n^M\|_\infty$	δ_{IM}
2	1.93×10^{-1}		1.27×10^{-2}		6.92×10^{-3}		3.30×10^{-4}	
4	1.08×10^{-1}	0.84	3.15×10^{-3}	2.01	1.02×10^{-3}	2.76	2.13×10^{-5}	3.95
8	5.73×10^{-2}	0.91	7.86×10^{-4}	2.00	1.40×10^{-4}	2.87	1.34×10^{-6}	3.99
16	2.93×10^{-2}	0.97	1.96×10^{-4}	2.00	1.82×10^{-5}	2.94	8.37×10^{-8}	4.00
32	1.45×10^{-2}	1.01	4.91×10^{-5}	2.00	2.27×10^{-6}	3.00	5.23×10^{-9}	4.00

As shown in Table 3.4.1, under smooth kernel conditions, the computed orders of convergence for both collocation and iterated collocation solutions match well with the theoretical findings. Similarly, this example confirms that the computed orders for modified collocation and iterated modified collocation solutions also match the theoretical results, thereby validating the effectiveness of higher order collocation methods and one step iteration in achieving accelerated convergence.

Example 8. *Consider*

$$x(s) - \int_0^1 \kappa(s,t)\,[g(t,x(t))]\,dt = \int_0^1 \kappa(s,t) f(t) dt, \quad 0 \le s \le 1,$$

where

$$\kappa(s,t) = \begin{cases} s(1-t), & 0 \le s \le t, \\ (1-s)t, & t \le s \le 1, \end{cases} \quad \textit{and} \quad g(t,u) = \frac{1}{1+t+u}$$

with $f(t)$ *so chosen that* $\varphi(s) = \dfrac{s(1-s)}{s+1}$ *is the exact solution.*
This integral equation arises from the boundary value problem (see [10])

$$x''(s) = g(s, x(s)) + f(s), \quad 0 < s < 1, \quad x(0) = x(1) = 0.$$

Nonlinear integral equations of this type frequently arise in the modeling of complex physical phenomena and in the reformulation of boundary value problems with nonlinear boundary conditions associated with differential equations (see [13, 28, 66]). Such type of integral equations are examples of Hammerstien integral equations. Let $\mathcal{X}_n$ denote the space of piecewise constant functions defined with respect to the given uniform partition of $[0,1]$. Define the interpolatory projection operator $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ as in equation (3.1.3), where the projection is taken at the midpoints of the subintervals. For numerical integration, we use the composite 2-point Gauss quadrature rule corresponding to the same partition. To obtain the order convergence of $\widetilde{\varphi}_n^M$ as $O(1/n^4)$, we apply composite 2-point Gauss quadrature rule with n^2 subintervals.

Table 3.4.2: Comparison of convergence rates in collocation variants for Green's kernel.

n	$\|\varphi - \varphi_n^C\|_\infty$	δ_C	$\|\varphi - \varphi_n^S\|_\infty$	δ_S	$\|\varphi - \varphi_n^M\|_\infty$	δ_M	$\|\varphi - \widetilde{\varphi}_n^M\|_\infty$	δ_{IM}
2	1.50×10^{-1}		1.30×10^{-3}		1.31×10^{-3}		1.31×10^{-3}	
4	9.54×10^{-2}	0.65	2.31×10^{-4}	2.49	1.68×10^{-4}	2.97	7.77×10^{-5}	4.07
6	6.87×10^{-2}	0.81	1.02×10^{-4}	2.01	5.71×10^{-5}	2.66	1.47×10^{-5}	4.10
8	5.34×10^{-2}	0.88	5.81×10^{-5}	1.95	2.62×10^{-5}	2.71	4.76×10^{-6}	3.92
10	4.35×10^{-2}	0.92	3.67×10^{-5}	2.07	1.37×10^{-5}	2.90	1.87×10^{-6}	4.19
12	3.66×10^{-2}	0.95	2.59×10^{-5}	1.90	8.22×10^{-6}	2.80	9.52×10^{-7}	3.69

In Table 3.4.2, a numerical example with Green's function-type kernel demonstrates that iterations improve the orders of convergence for both collocation and modified collocation solutions. In the collocation method improving from approximately first order to second order convergence upon iteration, and the modified collocation solution enhancing from third to nearly fourth order accuracy. This highlights the robustness of the iterated approaches, particularly the iterated modified collocation method, which maintains high convergence rates even in the presence of reduced kernel smoothness due to discontinuities along the diagonal.

3.5 Conclusion

In this chapter, we studied the collocation method and its variants for the numerical solution of Urysohn integral equations with both smooth kernels and non-smooth kernels of Green's function type. The discretization is based on interpolatory projections at $2r+1$ equidistant collocation points on each subinterval of a partition of $[0, 1]$, using even-degree piecewise polynomial spaces. While earlier studies show that Gauss points can improve convergence rates, it is generally observed that when non-Gaussian collocation points are employed, iteration does not enhance the order of convergence, irrespective of the kernel regularity, as illustrated in Example 6.

A key contribution of this chapter is the demonstration that higher-order convergence can however be achieved by employing even-degree piecewise polynomial spaces together with equidistant collocation points. This approach avoids the need for special node distributions while retaining strong theoretical convergence properties and competitive numerical performance. The results confirm that high-order accuracy is attainable within a simple and practical discretization framework, highlighting the effectiveness and flexibility of the proposed method. The main results of this chapter may be summarized as follows. For smooth kernels, the orders of convergence of the collocation, iterated collocation, modified collocation, and iterated modified collocation methods are h^{2r+1}, h^{2r+2}, h^{4r+3}, and h^{4r+4}, respectively. For Green's function-type kernels, the corresponding convergence orders are h^{2r+1}, h^{2r+2}, h^{2r+2}, and h^{2r+3}, whenever $r \geq 1$. In both cases, the iterated modified collocation method exhibits the fastest convergence, a fact that is also confirmed by the numerical results. Moreover, the iteration method leads to an improvement in the convergence rate of the corresponding collocation-based approximations. These findings further indicate the potential of extending the proposed framework to integral equations with singular or weakly singular kernels and to higher-dimensional problems such as boundary integral equations.

Chapter 4

Eigenvalue Problems for Compact Integral Operators [1]

This chapter is devoted to the analytical study of projection-based numerical methods for the approximation of eigenvalues and spectral subspaces associated with compact linear integral operators of the form $\mathcal{K}\psi = \lambda\psi$, where $\mathcal{K}$ is a compact linear integral operator acting on suitable Banach spaces over $[0, 1]$. The analysis focuses on integral operators with both smooth kernels and Green's function-type kernels. The numerical framework considered here is based on orthogonal and interpolatory projections onto finite-dimensional approximation spaces $\mathcal{X}_n$, consisting of piecewise polynomials of even degree $\leq 2r$ defined with respect to a uniform partition of the interval $[0, 1]$. The interpolatory projections are constructed using the earlier $2r+1$ equally spaced collocation points on each, which are not required to be Gauss points. Within this framework, standard and modified variants of Galerkin and collocation methods, together with their iterated versions, are investigated.

This chapter develops a systematic theory for the convergence of eigenvalue and spectral subspace approximations for both smooth and Green's function-type kernels. Higher-order convergence results are established for the modified and iterated projection methods, demonstrating significant improvements over the standard projection methods. Finally, numerical experiments are presented to illustrate and validate the theoretical findings obtained in this chapter.

[1] A portion of the work presented in this chapter has been published in Numerical Algorithms, 2026.

4.1 Preliminaries

Let $\mathcal{X} = \mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$, and $\Omega = [0,1] \times [0,1]$. Let κ be a real-valued continuous function on Ω. We aim to solve the eigenvalue problem of finding an eigenfunction $0 \neq \psi \in \mathcal{X}$ and an eigenvalue $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$\mathcal{K}\psi = \lambda\psi, \tag{4.1.1}$$

where $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is a compact linear integral operator, defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1], \quad x \in \mathcal{X}, \tag{4.1.2}$$

with the kernel function $\kappa \in \mathcal{C}(\Omega)$, that is, κ is continuous on the closed unit square.

The main objective is to compute nonzero approximate eigenvalues and the corresponding eigenfunctions of such operators. Efficient numerical methods for eigenvalue problems is essential in many engineering and scientific computing, see [13, 20]. For compact integral operators with smooth (or non-smooth) kernels, commonly used methods include Galerkin, collocation, degenerate kernel, and Nyström methods. Since compact operators have well-structured spectra consisting of isolated eigenvalues, projection-based numerical methods are particularly effective for this class of problems.

Let $\mathcal{K}_n$ be a sequence of continuous finite-rank operators converging pointwise to $\mathcal{K}$, and suppose the family $\{\mathcal{K}_n\}$ is collectively compact. The eigenvalue problem (4.1.1) is approximated by

$$\mathcal{K}_n\psi_n = \lambda_n\psi_n, \quad \lambda_n \in \mathbb{C} \setminus \{0\}, \quad 0 \neq \psi_n \in \mathcal{X}. \tag{4.1.3}$$

Since $\mathcal{K}_n$ is of finite rank, the eigenvalue problem (4.1.3) can be reduced to a matrix eigenvalue problem (see Atkinson [13] and Ahues et al. [1]). The approximate solutions (λ_n, ψ_n) converge to the exact solution (λ, ψ) under standard spectral approximation theory for compact operators. See [1, 13, 20] or Chapter 1.

When establishing the convergence of an approximate solution, a key consideration is often the rate at which the approximate solution converges to the exact solution. Several important methods have been developed for computing approximate solutions, including Nyström methods, where the integral in (4.1.2) is replaced by a convergent quadrature formula, and projection-based approaches such as Galerkin and collocation methods, along with their variants. The analysis of approximate solutions for eigenvalue problems with smooth kernels has been widely studied in the literature [1, 13, 15, 20, 86]. For kernels of the Green's function type, which often lack differentiability along the diagonal, early investigations were conducted in Chatelin-Lebbar [22], where interpolatory projections used Gaussian points (zeros of Legendre polynomials) as collocation nodes.

The improvement of spectral subspace approximations in projection methods through iterative techniques was first introduced by Sloan [84]. In Kulkarni [49, 52], a modified projection method was proposed for solving (4.1.1) using sequences of projections. It was shown that for sufficiently smooth kernels, the modified projection method yields faster convergence of eigenelements compared to the iterated projection method. Furthermore, convergence of eigenfunctions can be significantly enhanced using the iterated modified projection method, obtained from a one-step iteration of the modified projection approach. Other notable approaches include the degenerate kernel method developed by Nelakanti [63], which is based on kernel interpolation in both variables, provides error bounds for eigenvalues and the distance between spectral subspaces, and shows that iterating the eigenfunction approximation improves convergence. Nyström and degenerate kernel methods employing projections at Gauss points onto spaces of (discontinuous) piecewise polynomials have been shown to achieve high-order convergence for eigenvalue problems. In [2], superconvergent Nyström and degenerate kernel methods were proposed that enhance the convergence order for both eigenvalue and spectral subspace approximations when the kernel is sufficiently smooth, with further improvements in eigenfunction convergence via iteration techniques. These methods typically use piecewise polynomial basis functions, and the resulting finite-dimensional eigenvalue problems are solved in matrix form.

Consider a uniform partition of $[0, 1]$ given by $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$, where $t_j = \dfrac{j}{n}$ for $j = 0, 1, \ldots, n$, and let $h = \dfrac{1}{n}$. For an integer $r \geq 0$, let $\mathcal{X}_n$ denote the space of piecewise polynomials of even degree $\leq 2r$ associated with this partition. Chapter 1 establishes superconvergence for linear Fredholm integral equations with smooth and Green's kernels using standard and modified collocation methods based on our choice of simpler $2r + 1$ equidistant collocation points (not necessarily Gauss points) and even-degree polynomial spaces. In this chapter, we extend these results to the corresponding eigenvalue problems through orthogonal and interpolatory projection methods under the similar framework. We derive error estimates for eigenvalues and spectral subspaces, and show that an additional iteration step yields superconvergence of the eigenfunctions. Our results indicate that, for eigenvalues, the modified projection method achieves faster convergence than the standard projection method.

The structure of this chapter is as follows. Section 4.2 introduces spectral projections and presents the fundamental results useful throughout the chapter. Section 4.3 defines the orthogonal and interpolatory projection operators and establishes the key estimates required for the analysis of the projection-based approximation methods. Section 4.4 is devoted to the convergence analysis for integral operators with smooth kernels, where the main results concerning the orders of convergence of the proposed methods are derived. Section 4.5 extends the analysis to integral operators with Green's function-type kernels. Numerical illustrations validating the theoretical results are presented in Section 4.6. Finally, Section 4.7 summarizes the chapter with concluding remarks.

4.2 Spectral Projection

Let $\mathcal{X} = \mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$, and $\mathcal{BL}(\mathcal{X})$ will denote the set of all bounded linear operators on $\mathcal{X}$, equipped with the operator norm. Let $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ be a compact linear operator, and let the resolvent set of $\mathcal{K}$ is defined as

$$\rho(\mathcal{K}) = \{z \in \mathbb{C} : (\mathcal{K} - z\mathcal{I})^{-1} \in \mathcal{BL}(\mathcal{X})\}$$

and the spectrum of $\mathcal{K}$ is $\sigma(\mathcal{K}) = \mathbb{C}\backslash\rho(\mathcal{K})$. The point spectrum of $\mathcal{K}$ contains all $\lambda \in \sigma(\mathcal{K})$ for which $\mathcal{K} - \lambda\mathcal{I}$ is not injective. In such cases, λ is referred as an eigenvalue of $\mathcal{K}$. The geometric multiplicity of λ is defined as the dimension of the null space of $\mathcal{K} - \lambda\mathcal{I}$. Let Γ be a rectifiable, simple, closed, positively oriented curve contained in $\rho(\mathcal{K})$ that separates λ from the remainder of the spectrum of $\mathcal{K}$. The spectral projection corresponing to $\mathcal{K}$ and λ is then given by:

$$E = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{K} - z\mathcal{I})^{-1}\, dz \in \mathcal{BL}(\mathcal{X}),$$

and the dimension of $\mathcal{R}(E)$, the range of the operator E, is referred as the algebraic multiplicity of λ. If λ has finite algebraic multiplicity, it is classified as a spectral value of finite type. In this case, λ is an eigenvalue of $\mathcal{K}$. (See Ahues et al [1, Proposition 1.31].) Furthermore, if the algebraic multiplicity of λ is equal to one, λ is called a simple eigenvalue of $\mathcal{K}$. For any nonzero subspaces $\mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{X}$, define

$$\delta(\mathcal{Y}, \mathcal{Z}) = \sup\{d(y, \mathcal{Z}) : y \in \mathcal{Y}, \|y\| = 1\},$$

where $d(y, \mathcal{Z}) = \inf_{z\in\mathcal{Z}} \|y - z\|$. The gap between $\mathcal{Y}$ and $\mathcal{Z}$ is then given by

$$\hat{\delta}(\mathcal{Y}, \mathcal{Z}) = \max\{\delta(\mathcal{Y}, \mathcal{Z}), \delta(\mathcal{Z}, \mathcal{Y})\}.$$

This notion provides a quantitative measure of the distance between subspaces and will be used to study convergence of approximate eigenspaces.

Consider a sequence of operators $\{\mathcal{K}_n\} \subset \mathcal{BL}(\mathcal{X})$ that converges to $\mathcal{K}$ in a collectively compact manner, that is, $\mathcal{K}_n x \to \mathcal{K}x$ for all $x \in \mathcal{X}$ and the set $\{\mathcal{K}_n x : \|x\| \leq 1,\ n \in \mathbb{N}\}$.

As discussed in Chapter 1, for all sufficiently large n, the contour Γ is contained in the resolvent set $\rho(\mathcal{K}_n)$ and $\sup_{z\in\Gamma} \|(\mathcal{K}_n - z\mathcal{I})^{-1}\| < \infty$. Consequently, the spectral projection associated with $\mathcal{K}_n$, defined by

$$E_n = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{K}_n - z\mathcal{I})^{-1}\, dz,$$

is well defined and has rank m (see Propositions 10 and 11 in Chapter 1). In particular, the spectrum of $\mathcal{K}_n$ inside Γ consists of exactly m eigenvalues $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,m}$, counted according to their algebraic multiplicities. Define the arithmetic mean of these eigenvalues as

$$\hat{\lambda}_n = \frac{1}{m}\sum_{j=1}^{m} \lambda_{n,j}.$$

We now recall the following results from Chapter 1, which will be used in the subsequent analysis.

Proposition 28. *For all sufficiently large n, the gap between the spectral subspaces satisfies*

$$\hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n)\big) \leq C\big\|(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}\big\|,$$

where $C > 0$ is a constant independent of n, and $(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}$ denotes the restriction of $(\mathcal{K} - \mathcal{K}_n)\mathcal{K}$ to the spectral subspace $\mathcal{R}(E)$.

Proposition 29. *For all sufficiently large n, the arithmetic mean $\hat{\lambda}_n$ of the approximating eigenvalues satisfies*

$$|\lambda - \hat{\lambda}_n| \leq C\big\|\mathcal{K}_n(\mathcal{K} - \mathcal{K}_n)\mathcal{K}|_{\mathcal{R}(E)}\big\|.$$

The above results are due to Osborn [65] and are presented in the form used by Kulkarni [52]. Together, these propositions provide the fundamental error bounds for the approximation of the eigenelements, namely eigenvalues and spectral subspaces.

4.3 Projection Based Methods

Denote the uniform partition of $[0, 1]$ as

$$0 = t_0 < t_1 < \cdots < t_n = 1 \quad \text{where } t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n.$$

Denote $\Delta_j = [t_{j-1}, t_j]$. Let r be a non-negative integer, and define

$$\mathcal{X}_n = \left\{ x \in L^\infty[0,1] : x|_{\Delta_j} \text{ is a polynomial of even degree} \leq 2r,\ j = 1, 2, \ldots, n \right\}.$$

Let

$$h = t_j - t_{j-1} = \frac{1}{n}, \quad j = 1, 2, \ldots, n.$$

Since continuity at the partition points is not imposed, the approximation space $\mathcal{X}_n$ is a subspace of $\mathcal{L}^\infty[0,1]$. We consider two projection operators mapping into $\mathcal{X}_n$: the orthogonal projection $\mathcal{P}_n$, obtained as the restriction to $\mathcal{L}^\infty[0,1]$ of the $\mathcal{L}^2$-orthogonal projection onto $\mathcal{X}_n$, and the interpolatory projection $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$, defined by interpolation at $2r+1$ points on each subinterval. The construction and properties of these projections are briefly recalled in the following subsections; full details can be found in Chapter 1.

4.3.1 Orthogonal Projection

Let $\mathcal{X} = \mathcal{L}^2[0,1]$ denote the space of real-valued, square-integrable functions defined on the interval $[0,1]$, equipped with the standard inner product

$$\langle x, y \rangle = \int_0^1 x(t) y(t)\, dt.$$

For above uniform partition of $[0,1]$ with mesh size h, and $\mathcal{X}_n$ be the space of piecewise polynomials of even degree at most $2r$. Using scaled Legendre polynomials $\{L_\eta\}_{\eta=0}^{2r}$ on each subinterval $\Delta_j = [t_{j-1}, t_j]$, we define local basis functions $\{\psi_{j,\eta}\}$, which form an orthonormal basis of $\mathcal{X}_n$ in $\mathcal{L}^2[0,1]$. See Chapter 1.

For $f, g \in \mathcal{C}(\Delta_j)$, define the local inner product by

$$\langle f, g \rangle_{\Delta_j} = \int_{t_{j-1}}^{t_j} f(t) g(t)\, dt,$$

and the supremum norm by

$$\|f\|_{\Delta_j,\infty} = \max_{t \in [t_{j-1}, t_j]} |f(t)|.$$

Then the following bound holds:

$$|\langle f, g \rangle_{\Delta_j}| \leq h\, \|f\|_{\Delta_j,\infty} \|g\|_{\Delta_j,\infty}.$$

Let $\mathbb{P}_{2r,\Delta_j}$ denote the space of polynomials of degree $\leq 2r$ on Δ_j. The local orthogonal projection $\mathcal{P}_{n,j} : \mathcal{C}(\Delta_j) \to \mathbb{P}_{2r,\Delta_j}$ is given by

$$\mathcal{P}_{n,j} x = \sum_{\eta=0}^{2r} \langle x, \psi_{j,\eta} \rangle_{\Delta_j} \psi_{j,\eta},$$

and satisfies $\langle \mathcal{P}_{n,j}x, y\rangle_{\Delta_j} = \langle x, \mathcal{P}_{n,j}y\rangle_{\Delta_j}$, $\mathcal{P}_{n,j}^2 = \mathcal{P}_{n,j}$ and $\mathcal{P}_{n,j}\mathcal{P}_{n,i} = 0$ for $i \neq j$. Moreover, from (4.3.2) there exists a constant $C_1 > 0$, independent of h, such that

$$\|\mathcal{P}_{n,j}x\|_{\Delta_j,\infty} \leq C_1\|x\|_\infty.$$

The global orthogonal projection $\mathcal{P}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ is then defined by

$$\mathcal{P}_n x = \sum_{j=1}^{n} \mathcal{P}_{n,j}x. \tag{4.3.1}$$

By a Hahn-Banach extension argument, $\mathcal{P}_n$ extends to a bounded projection on $\mathcal{L}^\infty[0,1]$ satisfying

$$\|\mathcal{P}_n\| \leq C_1. \tag{4.3.2}$$

Finally, if $x \in \mathcal{C}^{2r+1}(\Delta_j)$, then the following standard approximation estimate holds (see Chapter 1):

$$\|(\mathcal{I} - \mathcal{P}_{n,j})x\|_{\Delta_j,\infty} \leq C_2\left\|x^{(2r+1)}\right\|_{\Delta_j,\infty} h^{2r+1}, \tag{4.3.3}$$

where C_2 is a constant independent of h. In fact, let $\alpha \geq 0$ be an integer and set $\beta = \min\{\alpha,\, 2r+1\}$. Then, for any $x \in \mathcal{C}^\alpha[0,1]$ one can see that

$$\|(\mathcal{I} - \mathcal{P}_n)x\|_\infty \leq C_2\left\|x^{(\beta)}\right\|_\infty h^\beta. \tag{4.3.4}$$

4.3.2 Interpolatory Projection

Let $\mathcal{X} = \mathcal{C}[0,1]$ be the space of continuous real valued function on the interval $[0,1]$, equipped with the supremum norm defined as $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$. In each sub-interval $[t_{j-1}, t_j]$, consider $2r+1$ equally spaced interpolation points as

$$\tau_j^i = t_{j-1} + \frac{ih}{2r}, \quad i = 0, 1, \ldots, 2r \ \text{ for } \ r \geq 1, \quad \text{and} \quad \tau_j = \frac{t_{j-1} + t_j}{2} \ \text{ for } \ r = 0.$$

Note that these interpolation nodes are equidistant in each subinterval. The interpolatory operator $\mathcal{Q}_n : \mathcal{X} \to \mathcal{X}_n$ is defined as

$$\mathcal{Q}_n x(\tau_j^i) = x(\tau_j^i), \quad i = 0, 1, 2, \ldots, 2r;\ j = 1, 2, \ldots, n. \tag{4.3.5}$$

Note that $\mathcal{Q}_n x \to x$ for all $x \in \mathcal{C}[0,1]$. By employing the Hahn-Banach theorem, $\mathcal{Q}_n$ can be extended to $\mathcal{L}^\infty[0,1]$, making $\mathcal{Q}_n : \mathcal{L}^\infty[0,1] \to \mathcal{X}_n$ a projection.

If $x \in \mathcal{C}^{2r+1}(\Delta_j)$, a standard result from Chapter 1 implies that

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq C_3\left\|x^{(2r+1)}\right\|_\infty h^{2r+1}, \tag{4.3.6}$$

where C_3 is a constant independent of h. More generally, for any $x \in \mathcal{C}^{\alpha}[0,1]$,

$$\|(\mathcal{I} - \mathcal{Q}_n)x\|_\infty \leq C_3 \left\|x^{(\beta)}\right\|_\infty h^\beta,$$

where $\beta = \min\{\alpha,\, 2r+1\}$ with $\alpha \geq 0$ be an integer.

4.3.3 Estimates in the Methods of Approximation

The classical projection method for approximating the eigenvalue problem (4.1.1) is obtained by replacing $\mathcal{K}_n$ by $\pi_n\mathcal{K}$ in (4.1.3) as

$$\pi_n\mathcal{K}\psi_n = \lambda_n\psi_n, \quad \lambda_n \in \mathbb{C}\setminus\{0\},\ \psi_n \in \mathcal{X}_n,\ \|\psi_n\|_\infty = 1.$$

A one-step iteration to refine the eigenfunction is given by

$$\psi_n^S = \frac{1}{\lambda_n}\mathcal{K}\psi_n,$$

which satisfies the condition

$$\mathcal{K}\pi_n\psi_n^S = \lambda_n\psi_n^S.$$

In the modified projection method, the operator $\mathcal{K}_n$ is replaced with a finite-rank operator defined as

$$\mathcal{K}_n^M = \pi_n\mathcal{K}\pi_n + \pi_n\mathcal{K}(\mathcal{I}-\pi_n) + (\mathcal{I}-\pi_n)\mathcal{K}\pi_n = \pi_n\mathcal{K} + \mathcal{K}\pi_n - \pi_n\mathcal{K}\pi_n.$$

As a consequence,

$$\left\|\mathcal{K} - \mathcal{K}_n^M\right\| = \|(\mathcal{I}-\pi_n)\mathcal{K}(\mathcal{I}-\pi_n)\| \to 0 \quad \text{as } n \to \infty.$$

The approximation (4.1.3) now requires finding $\lambda_n^M \in \mathbb{C}\setminus\{0\}$ and $\psi_n^M \in \mathcal{X}$ such that $\left\|\psi_n^M\right\|_\infty = 1$ and

$$\mathcal{K}_n^M\psi_n^M = \lambda_n^M\psi_n^M.$$

The corresponding iterated eigenfunction for the modified method is defined as

$$\widetilde{\psi}_n^M = \frac{1}{\lambda_n^M}\mathcal{K}\psi_n^M.$$

Let $\{\mathcal{K}_n\}$ be a sequence of finite rank bounded linear operators converging to $\mathcal{K}$ in the collectively compact sense. We consider the standard and modified projection-based approximations defined by $\mathcal{K}_n = \pi_n\mathcal{K}$ and $\mathcal{K}_n = \mathcal{K}_n^M$, respectively. Let λ be a nonzero eigenvalue of $\mathcal{K}$ with algebraic multiplicity m, and let $\hat{\lambda}_n$ and $\hat{\lambda}_n^M$ denote the arithmetic means of the m eigenvalues of $\pi_n\mathcal{K}$ and $\mathcal{K}_n^M$, respectively, lying inside a simple closed contour Γ that encloses λ. Let E, E_n, and E_n^M denote the corresponding spectral projections associated with Γ. The associated spectral subspaces are denoted by $\mathcal{R}(E)$, $\mathcal{R}(E_n)$, and $\mathcal{R}(E_n^M)$. Then, the results presented below are consequences of Propositions 28 and 29 and are stated as Theorems 12 and 13 in Chapter 1.

For any $\psi_n \in \mathcal{R}(E_n)$, define $\psi_n^S = \mathcal{K}\psi_n$. Then, for all sufficiently large n,

$$\|\psi_n - E\psi_n\| \leq \hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n)\big) \leq C\big\|(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|, \tag{4.3.7}$$

$$\big\|\psi_n^S - E\psi_n^S\big\| \leq \delta\big(\mathcal{R}(E), \mathcal{K}\mathcal{R}(E_n)\big) \leq C\big\|\mathcal{K}(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|, \tag{4.3.8}$$

and

$$|\lambda - \hat{\lambda}_n| \leq C\big\|\mathcal{K}(\mathcal{K} - \pi_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\|. \tag{4.3.9}$$

For any $\psi_n^M \in \mathcal{R}(E_n^M)$, define $\widetilde{\psi}_n^M = \mathcal{K}\psi_n^M$. Then, for all sufficiently large n,

$$\big\|\psi_n^M - E\psi_n^M\big\| \leq \hat{\delta}\big(\mathcal{R}(E), \mathcal{R}(E_n^M)\big) \leq C\big\|(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\big\|, \tag{4.3.10}$$

$$\Big\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\Big\| \leq \delta\big(\mathcal{R}(E), \mathcal{K}\mathcal{R}(E_n^M)\big) \leq C\big\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\big\|, \tag{4.3.11}$$

and

$$|\lambda - \hat{\lambda}_n^M| \leq C\big\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\big\|. \tag{4.3.12}$$

Throughout this chapter, we restrict our attention to the case in which λ is a simple eigenvalue of $\mathcal{K}$. For the projection-based schemes introduced earlier, we take $\pi_n = \mathcal{P}_n$, which yields the Galerkin method and its associated variants. Similarly, choosing $\pi_n = \mathcal{Q}_n$ leads to the collocation method and its variants. Under these settings, we establish the main convergence results for both the Galerkin and collocation methods and their variants. In the subsequent sections, we investigate the convergence of eigenelements for compact integral operators with both smooth kernels and Green's function-type kernels.

4.4 Convergence Analysis for Smooth Kernels

Consider the Fredholm integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where the kernel function κ is assumed to be smooth, that is, $\kappa \in \mathcal{C}^{2r+2}(\Omega)$. Throughout this section, we adopt the following notation: let j and k be non-negative integers such that $0 \leq j+k \leq 2r+3$. We denote the smoothness of kernel κ in mixed partial derivatives as

$$D^{(j,k)}\kappa(s,t) = \frac{\partial^{j+k}\kappa}{\partial s^j \partial t^k}(s,t).$$

Then, define the constant

$$C_4 = \max_{0\leq j+k\leq 2r+3} \left\{ \sup_{0\leq s,t\leq 1} \left| D^{(j,k)}\kappa(s,t) \right| \right\}.$$

For $x \in \mathcal{C}[0,1]$ and $0 \leq j \leq 2r+2$, we have

$$(\mathcal{K}x)^{(j)}(s) = \int_0^1 \frac{\partial^j \kappa}{\partial s^j}(s,t)\, x(t)\, dt = \int_0^1 D^{(j,0)}\kappa(s,t)\, x(t)\, dt, \quad s \in [0,1].$$

Hence,

$$\left\| (\mathcal{K}x)^{(j)} \right\|_\infty \leq C_4 \|x\|_\infty. \tag{4.4.1}$$

4.4.1 Orthogonal Projection Based Approximations

In this subsection, we present some key results that are essential for the approximation of eigenelements using orthogonal projections. Although several of these results are already available in the literature, we state them here for completeness and for ease of reference in the subsequent analysis.

Proposition 30. *If $x \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq (C_2)^2 C_4 \left\| x^{(2r+1)} \right\|_\infty h^{4r+2}.$$

The above result is adapted from Chatelin-Lebbar [21, Theorem 3]. A related following estimate can also be found in Kulkarni [48, Proposition 4.2].

Proposition 31. *If $x \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_5 \left\| x^{(2r+1)} \right\|_\infty h^{6r+3},$$

and

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_6 \left\| x^{(2r+1)} \right\|_\infty h^{8r+4},$$

where $C_5 = (C_2)^3 C_4$ and $C_6 = \left((C_2)^2 C_4\right)^2$, are the constants independent of h.

Lemma 4.4.1. *Let $\mathcal{K}$ be the linear integral operator defined by (4.1.2) with a smooth kernel $\kappa \in \mathcal{C}^{2r+1}(\Omega)$. Let $\mathcal{P}_n$ be the orthogonal projection onto the approximating space $\mathcal{X}_n$ defined by (4.3.1). Then*

$$\begin{aligned} \left\| (\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)} \right\| &= O\left(h^{2r+1}\right), \\ \left\| \mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)} \right\| &= O\left(h^{4r+2}\right). \end{aligned}$$

Proof. By definition of the operator norm,

$$\left\| (\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)} \right\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{ \|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1 \Big\}.$$

Let $\psi \in \mathcal{R}(E)$ and set $y = \mathcal{K}\psi$. Since $\kappa \in \mathcal{C}^{2r+1}(\Omega)$, differentiation under the integral sign is valid up to order $2r+1$. Hence, $\mathcal{K}$ maps $\mathcal{C}[0,1]$ continuously into $\mathcal{C}^{2r+1}[0,1]$, so that $y \in \mathcal{C}^{2r+1}[0,1]$, and consequently $\mathcal{K}y \in \mathcal{C}^{2r+2}[0,1]$. Then

$$(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi = (\mathcal{I} - \mathcal{P}_n)\mathcal{K}y.$$

Using the estimate (4.3.4) of the orthogonal projection, we obtain

$$\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}y\|_\infty \le C_2 \left\| (\mathcal{K}y)^{(2r+1)} \right\|_\infty h^{2r+1}.$$

From inequality (4.4.1), we have

$$\left\| (\mathcal{K}y)^{(2r+1)} \right\|_\infty \le C_4 \|y\|_\infty \le (C_4)^2 \|\psi\|_\infty.$$

Therefore, we obtain

$$\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty \le C_2 (C_4)^2 \|\psi\|_\infty h^{2r+1}.$$

Since $\|\psi\|_\infty = 1$, we conclude that

$$\left\| (\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)} \right\| = O(h^{2r+1}).$$

Next, we estimate

$$\left\| \mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)} \right\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{ \|\mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1 \Big\}.$$

Let $\psi \in \mathcal{C}^{2r+1}[0,1]$. Then $y = \mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$, and consequently $\mathcal{K}y \in \mathcal{C}^{2r+1}[0,1]$. Therefore,

$$\mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi = \mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}y.$$

Applying Proposition 30, we obtain

$$\|\mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty = \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}y\|_\infty \le (C_2)^2 C_4 \left\| (\mathcal{K}y)^{(2r+1)} \right\|_\infty h^{4r+2}.$$

As shown earlier, the quantity $\left\| (\mathcal{K}y)^{(2r+1)} \right\|_\infty$ is independent of h and so

$$\|\mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty \le (C_2)^2 C_4 \|\psi\|_\infty h^{4r+2},$$

which implies

$$\left\| \mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)} \right\| = O\big(h^{4r+2}\big).$$

This completes the proof. □

Lemma 4.4.2. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a smooth kernel $\kappa \in \mathcal{C}^{2r+1}(\Omega)$. Let $\mathcal{P}_n$ be the orthogonal projection onto the approximating space $\mathcal{X}_n$ defined by* (4.3.1)*. Then*

$$\begin{aligned}\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| &= O\big(h^{6r+3}\big)\,,\\ \left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| &= O\big(h^{8r+4}\big)\,,\end{aligned}$$

where $\mathcal{K}_n^M = \mathcal{P}_n\mathcal{K} + \mathcal{K}\mathcal{P}_n - \mathcal{P}_n\mathcal{K}\mathcal{P}_n$.

Proof. Let $\psi \in \mathcal{R}(E)$. Since $\mathcal{R}(E)$ is invariant under $\mathcal{K}$, we have $\mathcal{K}\psi \in \mathcal{R}(E)$. By definition of the operator norm restricted to $\mathcal{R}(E)$,

$$\begin{aligned}\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| &= \sup_{\psi\in\mathcal{R}(E)}\Big\{\big\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\psi\big\|_\infty : \|\psi\|_\infty = 1\Big\}\\ &= \sup_{\psi\in\mathcal{R}(E)}\Big\{\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.\end{aligned}$$

Let $\psi \in \mathcal{C}^{2r+1}[0,1]$. Then $\mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$. Applying Proposition 31, we obtain

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \le C_5\big\|(\mathcal{K}\psi)^{(2r+1)}\big\|_\infty h^{6r+3}.$$

Further, by inequality (4.4.1),

$$\big\|(\mathcal{K}\psi)^{(2r+1)}\big\|_\infty \le C_4\|\psi\|_\infty.$$

Hence,

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \le C_4C_5\|\psi\|_\infty h^{6r+3}.$$

Since $\|\psi\|_\infty = 1$, it follows that

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = O\big(h^{6r+3}\big)\,.$$

Next, we estimate

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)}\Big\{\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

Again, since $\psi \in \mathcal{C}^{2r+1}[0,1]$, we have $\mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$. Applying Proposition 31, we obtain

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \le C_6\big\|(\mathcal{K}\psi)^{(2r+1)}\big\|_\infty h^{8r+4}.$$

Using again inequality (4.4.1), we get

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \le C_4C_6\|\psi\|_\infty h^{8r+4}.$$

Therefore, for $\|\psi\|_\infty = 1$, we conclude that

$$\left\| \mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)} \right\| = O\big(h^{8r+4}\big)\,.$$

This completes the proof. □

By combining (4.3.7)-(4.3.9) with Lemma 4.4.1, we obtain the following convergence rates for the approximation of eigenvalues and spectral subspaces by the Galerkin and iterated Galerkin methods.

Theorem 32. *Let λ be a simple eigenvalue of $\mathcal{K}$, and let λ_n^G denote the corresponding Galerkin eigenvalue of $\mathcal{P}_n\mathcal{K}$. Let $\psi_n \in \mathcal{R}(E_n)$ be a Galerkin eigenfunction, and let ψ_n^S denote the associated iterated Galerkin eigenfunction. Then, for sufficiently large n, the following estimates hold:*

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \big\|\psi_n^S - E\psi_n^S\big\|_\infty = O\big(h^{4r+2}\big)\,,$$

and

$$|\lambda - \lambda_n^G| = O\big(h^{4r+2}\big)\,.$$

In a similar manner, by combining (4.3.10)-(4.3.12) with Lemma 4.4.2, we obtain the following convergence rates for the modified and iterated modified Galerkin methods.

Theorem 33. *Let λ be a simple eigenvalue of $\mathcal{K}$, and let λ_n^M denote the corresponding modified Galerkin eigenvalue of $\mathcal{K}_n^M$. Let $\psi_n^M \in \mathcal{R}(E_n^M)$ be a modified Galerkin eigenfunction, and let $\widetilde{\psi}_n^M$ denote the associated iterated modified Galerkin eigenfunction. Then, for sufficiently large n, the following estimates hold:*

$$\big\|\psi_n^M - E\psi_n^M\big\|_\infty = O\big(h^{6r+3}\big)\,, \quad \left\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\right\|_\infty = O\big(h^{8r+4}\big)\,,$$

and

$$|\lambda - \lambda_n^M| = O\big(h^{8r+4}\big)\,.$$

The above theorems show that the standard Galerkin method yields eigenfunction and eigenvalue approximations of orders $O(h^{2r+1})$ and $O(h^{4r+2})$, respectively, while the iterated Galerkin method improves the eigenfunction accuracy to $O(h^{4r+2})$. The modified Galerkin scheme further enhances the convergence behaviour, producing eigenvalue errors of order $O(h^{8r+4})$ and eigenfunction errors of order $O(h^{6r+3})$, whereas the iterated modified Galerkin method achieves the highest accuracy, namely $O(h^{8r+4})$ for eigenfunctions. These results clearly demonstrate the superconvergence effect obtained through modified projection and iteration.

4.4.2 Interpolatory Projection Based Approximations

We now recall several results from Chapter 2 that will play a crucial role in the approximation of eigenelements using the interpolatory projection method. For completeness, we restate below Propositions 14-16 from Chapter 2.

Proposition 34. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 2C_4 \|x\|_{2r+2,\infty}\, h^{2r+2}.$$

Proposition 35. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le C_7 \|x\|_{2r+2,\infty}\, h^{4r+3},$$

where C_7 is a constant independent of h.

Proposition 36. *If $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)x\|_\infty \le 4(C_4)^2 \|x\|_{2r+2,\infty}\, h^{4r+4}.$$

Propositions 34-36 provide the fundamental approximation estimates for the interpolatory projection operator $\mathcal{Q}_n$, which will be instrumental in deriving the convergence rates for the corresponding eigenvalue and eigenfunction approximations.

Lemma 4.4.3. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a smooth kernel $\kappa \in \mathcal{C}^{2r+1}(\Omega)$ and continuously differentiable with respect to t. Let $\mathcal{Q}_n$ denote the interpolatory projection operator onto the approximation space $\mathcal{X}_n$ as defined in* (4.3.5). *Then*

$$\begin{aligned} \big\|(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\| &= O\big(h^{2r+1}\big)\,, \\ \big\|\mathcal{K}(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\big\| &= O\big(h^{2r+2}\big)\,. \end{aligned}$$

Proof. By definition of the operator norm,

$$\Big\|(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)}\Big\| = \sup_{\psi\in\mathcal{R}(E)} \Big\{\|(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

Let $\psi \in \mathcal{R}(E)$ and set $y = \mathcal{K}\psi$. Since $\kappa \in \mathcal{C}^{2r+1}(\Omega)$, differentiation under the integral sign is valid up to order $2r+1$. Hence, $\mathcal{K} : \mathcal{C}[0,1] \to \mathcal{C}^{2r+1}[0,1]$, so that $y \in \mathcal{C}^{2r+1}[0,1]$ and consequently $\mathcal{K}y \in \mathcal{C}^{2r+2}[0,1]$. Then using estimate (4.3.6),

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}y\|_\infty \le C_3 \big\|(\mathcal{K}y)^{(2r+1)}\big\|_\infty\, h^{2r+2}.$$

From inequality (4.4.1),

$$\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty \leq (C_4)^2\|\psi\|_\infty.$$

Hence,

$$\|(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty \leq (C_4)^2 \, \|\psi\|_\infty \, h^{2r+2},$$

and therefore

$$\left\|(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+2}\big)\,.$$

Next,

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)}\Big\{\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}y\|_\infty : \ \|\psi\|_\infty = 1\Big\}.$$

Applying Proposition 34,

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}y\|_\infty \leq 2C_4\|\mathcal{K}y\|_{2r+2,\infty}\, h^{4r+4}.$$

Note that

$$\|\mathcal{K}y\|_{2r+2,\infty} = \max_{0\leq j\leq 2r+2}\left\|(\mathcal{K}y)^{(j)}\right\|_\infty.$$

From inequality (4.4.1), it follows that the bound $\left\|(\mathcal{K}y)^{(j)}\right\|_\infty$ is independent of h. Thus,

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{Q}_n\mathcal{K})\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = O\big(h^{4r+4}\big)\,.$$

This completes the proof. □

Lemma 4.4.4. *Let $\mathcal{K}$ be the linear integral operator defined in* (4.1.2) *with a smooth kernel $\kappa \in \mathcal{C}^{2r+2}[0,1]$. Let $\mathcal{Q}_n$ denote the interpolatory projection operator onto the approximation space $\mathcal{X}_n$, as given by* (4.3.5). *Then*

$$\begin{aligned}
\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| &= O\big(h^{4r+3}\big)\,,\\
\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| &= O\big(h^{4r+4}\big)\,,
\end{aligned}$$

where $\mathcal{K}_n^M = \mathcal{Q}_n\mathcal{K} + \mathcal{K}\mathcal{Q}_n - \mathcal{Q}_n\mathcal{K}\mathcal{Q}_n$.

Proof. Let $\psi \in \mathcal{R}(E)$. Since $\mathcal{R}(E)$ is invariant under $\mathcal{K}$, it follows that $\mathcal{K}\psi \in \mathcal{R}(E)$. By definition of the operator norm restricted to $\mathcal{R}(E)$,

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)}\Big\{\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty : \ \|\psi\|_\infty = 1\Big\}.$$

Assume $\psi \in \mathcal{C}^{2r+2}[0,1]$ and $\kappa \in \mathcal{C}^{2r+2}(\Omega)$. Then $\mathcal{K}\psi \in \mathcal{C}^{2r+2}[0,1]$. Applying Proposition 35, we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq C_7 \|\mathcal{K}\psi\|_{2r+2,\infty}\, h^{4r+3}.$$

Moreover,

$$\|\mathcal{K}\psi\|_{2r+2,\infty} = \max_{0\leq j\leq 2r+2} \left\|(\mathcal{K}\psi)^{(j)}\right\|_\infty \leq C_4\|\psi\|_\infty,$$

so that

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq C_4 C_7 \|\psi\|_\infty\, h^{4r+3}.$$

Taking $\|\psi\|_\infty = 1$, we conclude

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = O\big(h^{4r+3}\big)\,.$$

Next, consider

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)} \Big\{\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty :\ \|\psi\|_\infty = 1\Big\}.$$

Since $\mathcal{K}\psi \in \mathcal{C}^{2r+2}[0,1]$, Proposition 36 yields

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq 4(C_4)^2 \|\mathcal{K}\psi\|_{2r+2,\infty}\, h^{4r+4}.$$

Using again the bound $\|\mathcal{K}\psi\|_{2r+2,\infty} \leq C_4\|\psi\|_\infty$, we obtain

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| \leq 4(C_4)^3 \|\psi\|_\infty\, h^{4r+4}.$$

Hence, for $\|\psi\|_\infty = 1$,

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}\big|_{\mathcal{R}(E)}\right\| = O\big(h^{4r+4}\big)\,.$$

This completes the proof. □

By combining the estimates from (4.3.7), (4.3.8), (4.3.9) along with Lemma 4.4.3, we obtain the following convergence rates for the approximation of eigenvalues and spectral subspaces using the collocation and iterated collocation methods.

Theorem 37. *Let $\psi_n \in \mathcal{R}(E_n)$ denote a collocation eigenfunction, and let ψ_n^S be the corresponding iterated collocation eigenfunction. Suppose λ_n^C is the approximate simple eigenvalue of $\mathcal{Q}_n\mathcal{K}$. Then, for sufficiently large n, the following error bounds hold:*

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \left\|\psi_n^S - E\psi_n^S\right\|_\infty = O\big(h^{2r+2}\big)\,,$$

and

$$|\lambda - \lambda_n^C| = O\big(h^{2r+2}\big)\,.$$

Similarly, using the results from (4.3.10), (4.3.11), (4.3.12) together with Lemma 4.4.4, we derive the following convergence estimates for the modified collocation and iterated modified collocation methods.

Theorem 38. *Let $\psi_n^M \in \mathcal{R}(E_n^M)$ be a modified collocation eigenfunction, and let $\widetilde{\psi}_n^M$ denote the corresponding iterated modified collocation eigenfunction. Suppose λ_n^M is the approximate simple eigenvalue of $\mathcal{K}_n^M$ obtained from the modified method. Then, for all sufficiently large n, the following convergence orders are satisfied:*

$$\left\|\psi_n^M - E\psi_n^M\right\|_\infty = O\big(h^{4r+3}\big)\,, \quad \left\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\right\|_\infty = O\big(h^{4r+4}\big)\,,$$

and

$$|\lambda - \lambda_n^M| = O\big(h^{4r+4}\big)\,.$$

The above results show that the standard collocation method produces eigenfunction and eigenvalue approximations of orders $O(h^{2r+1})$ and $O(h^{2r+2})$, respectively, while the iterated collocation method improves the eigenfunction accuracy by one additional order. In contrast, the modified collocation method significantly enhances the convergence behaviour, yielding eigenvalue approximations of order $O(h^{4r+4})$, with the iterated modified method attaining the highest accuracy.

These findings confirm that modified projection and iteration are highly effective strategies for achieving high-accuracy approximations of eigenelements, even when simple equidistant collocation points are employed on each subinterval of a uniform partition over an even-degree piecewise polynomial approximation space.

4.5 Convergence Analysis for Green's Kernels

Let $\mathcal{X} = \mathcal{L}^2[0,1]$ or $\mathcal{C}[0,1]$ and recall that the compact linear integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ is defined as

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\,dt, \quad s \in [0,1], \quad x \in \mathcal{X},$$

where $\kappa(s,t)$ is a real-valued continuous kernel function of the Green's function type. Consider the partition of Ω as

$$\Omega_1 = \{(s,t) \mid 0 \le t \le s \le 1\}, \quad \Omega_2 = \{(s,t) \mid 0 \le s \le t \le 1\}.$$

Suppose there exist functions $\kappa_i \in \mathcal{C}^\alpha(\Omega_i)$, $i = 1, 2$, for some integer $\alpha > r \geq 0$, such that the kernel κ can be written as

$$\kappa(s,t) = \begin{cases} \kappa_1(s,t), & (s,t) \in \Omega_1, \\ \kappa_2(s,t), & (s,t) \in \Omega_2, \end{cases}$$

and $\kappa_1(s,s) = \kappa_2(s,s),$ for all $s \in [0,1]$. A kernel κ satisfying these properties is referred to as a Green's kernel. Such kernels are generally non-smooth, as their regularity is restricted to the separate domains Ω_1 and Ω_2, with only continuity across the diagonal.

Here, we will fix some notations to be used throughout the section:
Let j, k be non-negative integer and we set

$$D^{(j,k)}\kappa_i(s,t) = \frac{\partial^{j+k}\kappa_i}{\partial s^j \partial t^k}(s,t), \quad \text{for } i = 1, 2.$$

Define

$$C_8 = \max_{0 \leq j,k \leq 2r+2} \left\{ \sup_{(s,t)\in\Omega_i} \left| D^{(j,k)}\kappa_i(s,t) \right| : \text{ for } i = 1, 2 \right\},$$

Now, for $x \in \mathcal{C}[0,1]$ and $s \in [0,1]$, we have

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)x(t)\ dt = \int_0^s \kappa_1(s,t)x(t)\ dt + \int_s^1 \kappa_2(s,t)x(t)\ dt.$$

Using the Leibnitz rule

$$(\mathcal{K}x)'(s) = \int_0^s \frac{\partial \kappa_1}{\partial s}(s,t)x(t)\ dt + \int_s^1 \frac{\partial \kappa_2}{\partial s}(s,t)x(t)\ dt,$$

and

$$(\mathcal{K}x)''(s) = \left(\frac{\partial \kappa_1}{\partial s}(s,s) - \frac{\partial \kappa_2}{\partial s}(s,s) \right) x(s) + \int_0^s \frac{\partial^2 \kappa_1}{\partial s^2}(s,t)x(t)\ dt + \int_s^1 \frac{\partial^2 \kappa_2}{\partial s^2}(s,t)x(t)\ dt,$$

which implies

$$|(\mathcal{K}x)'(s)| \leq \left(\sup_{(s,t)\in\Omega_1} \left| D^{(1,0)}\kappa_1(s,t) \right| \right) s\|x\|_\infty + \left(\sup_{(s,t)\in\Omega_2} \left| D^{(1,0)}\kappa_2(s,t) \right| \right) (1-s)\|x\|_\infty,$$

and

$$\begin{aligned} |(\mathcal{K}x)''(s)| \leq & \left(\sup_{(s,t)\in\Omega_1} \left| D^{(1,0)}\kappa_1(s,t) \right| + \sup_{(s,t)\in\Omega_2} \left| D^{(1,0)}\kappa_2(s,t) \right| \right) \|x\|_\infty \\ & + \left(\sup_{(s,t)\in\Omega_1} \left| D^{(2,0)}\kappa_1(s,t) \right| \right) s\|x\|_\infty + \left(\sup_{(s,t)\in\Omega_2} \left| D^{(2,0)}\kappa_2(s,t) \right| \right) (1-s)\|x\|_\infty. \end{aligned}$$

Then,

$$\left\|(\mathcal{K}x)^{(j)}\right\|_\infty \leq C_8 \|x\|_\infty, \quad \text{for } j = 0, 1, 2. \tag{4.5.1}$$

Let $x \in \mathcal{C}^2[0,1]$, we have

$$\begin{aligned}(\mathcal{K}x)^{(3)}(s) &= 2\left(\frac{\partial^2 \kappa_1}{\partial s^2}(s,s) - \frac{\partial^2 \kappa_2}{\partial s^2}(s,s)\right)x(s) + \left(\frac{\partial \kappa_1}{\partial s}(s,s) - \frac{\partial \kappa_2}{\partial s}(s,s)\right)x'(s) \\ &\quad + \int_0^s \frac{\partial^3 \kappa_1}{\partial s^3}(s,t)x(t)\, dt + \int_s^1 \frac{\partial^3 \kappa_2}{\partial s^3}(s,t)x(t)\, dt,\end{aligned}$$

and

$$\begin{aligned}(\mathcal{K}x)^{(4)}(s) &= 3\left(\frac{\partial^3 \kappa_1}{\partial s^3}(s,s) - \frac{\partial^3 \kappa_2}{\partial s^3}(s,s)\right)x(s) \\ &\quad + 3\left(\frac{\partial^2 \kappa_1}{\partial s^2}(s,s) - \frac{\partial^2 \kappa_2}{\partial s^2}(s,s)\right)x'(s) + \left(\frac{\partial \kappa_1}{\partial s}(s,s) - \frac{\partial \kappa_2}{\partial s}(s,s)\right)x''(s) \\ &\quad + \int_0^s \frac{\partial^4 \kappa_1}{\partial s^4}(s,t)x(t)\, dt + \int_s^1 \frac{\partial^4 \kappa_2}{\partial s^4}(s,t)x(t)\, dt.\end{aligned}$$

This implies that, if $x \in \mathcal{C}^2[0,1]$, then

$$\left\|(\mathcal{K}x)^{(3)}\right\|_\infty \leq C_8\left(\|x\|_\infty + \|x'\|_\infty\right),$$

$$\left\|(\mathcal{K}x)^{(4)}\right\|_\infty \leq C_8\left(\|x\|_\infty + \|x'\|_\infty + \|x''\|_\infty\right).$$

Proceeding similar way, if $x \in \mathcal{C}^{2r}[0,1]$ for $r \geq 1$ and $\kappa_i \in \mathcal{C}^\alpha(\Omega_i)$, $i = 1, 2$, for some integer $\alpha > r \geq 0$ then

$$\left\|(\mathcal{K}x)^{(2r+1)}\right\|_\infty \leq C_8\left(\|x\|_\infty + \|x'\|_\infty + \cdots + \left\|x^{(2r-1)}\right\|_\infty\right), \tag{4.5.2}$$

$$\left\|(\mathcal{K}x)^{(2r+2)}\right\|_\infty \leq C_8\left(\|x\|_\infty + \|x'\|_\infty + \cdots + \left\|x^{(2r)}\right\|_\infty\right). \tag{4.5.3}$$

Recall that, we are dealing with λ as a simple eigenvalue of $\mathcal{K}$, and we consider projection-based schemes with $\pi_n = \mathcal{P}_n$ and $\pi_n = \mathcal{Q}_n$, corresponding to the orthogonal and interpolatory projections, respectively. Under these choices, we establish convergence results for the associated eigenvalue approximations in upcoming subsections.

4.5.1 Orthogonal Projection Based Approximations

In this subsection, we present key results that are essential for the approximation of eigenelements using orthogonal projection.

Proposition 39. *For $r \geq 1$, if $x \in \mathcal{C}^{2r+1}[0,1]$, then*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_9 \left\|x^{(2r+1)}\right\|_\infty h^{2r+3},$$

where C_9 is some constant independent of h.
In particular, if $x \in \mathcal{C}[0,1]$, then

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_{10}\|x\|_\infty h^2,$$

for some constant C_{10} independent of h.

Proof. First, we prove the case $r \geq 1$. Let $s \in [0,1]$ and define $\kappa_s(t) = \kappa(s,t)$. Then, consider

$$\begin{aligned}
\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s) &= \int_0^1 \kappa_s(t)(\mathcal{I}-\mathcal{P}_n)x(t)\,dt \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \kappa_s(t)(\mathcal{I}-\mathcal{P}_{n,j})x(t)\,dt \\
&= \sum_{j=1}^n \langle \kappa_s, (\mathcal{I}-\mathcal{P}_{n,j})x\rangle_{\Delta_j} \\
&= \sum_{j=1}^n \langle (\mathcal{I}-\mathcal{P}_{n,j})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,j})x\rangle_{\Delta_j}. \qquad (4.5.4)
\end{aligned}$$

We analyze the two cases separately:
Case I: Let $s \in (t_{i-1}, t_i)$, for $i = 1, 2, \ldots, n$. Thus, the above expression can be rewritten as

$$\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s) = \sum_{\substack{j=1\\ j\neq i}}^n \langle (\mathcal{I}-\mathcal{P}_{n,j})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,j})x\rangle_{\Delta_j} + \langle (\mathcal{I}-\mathcal{P}_{n,i})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,i})x\rangle_{\Delta_i}. \quad (4.5.5)$$

For $j \neq i$, we note that both $\kappa_s, x \in \mathcal{C}^{2r+1}(\Delta_j)$. Thus, applying the estimate (4.3.3), we have

$$\left| \sum_{\substack{j=1\\ j\neq i}}^n \langle (\mathcal{I}-\mathcal{P}_{n,j})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,j})x\rangle_{\Delta_j} \right| \leq (C_2)^2 C_8 \left\|x^{(2r+1)}\right\|_\infty (n-1)h^{4r+3}. \qquad (4.5.6)$$

Now, for the case $j = i$, observe that κ_s is only continuous on Δ_i. We define a constant function as

$$\gamma_i(t) = \kappa_s(s), \quad t \in \Delta_i.$$

Then, we rewrite the term as

$$\langle (\mathcal{I}-\mathcal{P}_{n,i})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,i})x\rangle_{\Delta_i} = \langle \kappa_s - \gamma_i, (\mathcal{I}-\mathcal{P}_{n,i})x\rangle_{\Delta_i}.$$

For $t \in [t_{i-1}, t_i]$, we have

$$\kappa_s(t) - \gamma_i(t) = \begin{cases} D^{(0,1)}\kappa_1(s,\nu_1)(t-s), & \nu_1 \in (t,s), \\ D^{(0,1)}\kappa_2(s,\nu_2)(t-s), & \nu_2 \in (s,t), \end{cases}$$

by the mean value theorem. Further, using the bound (4.3.3), we get

$$|\langle (\mathcal{I}-\mathcal{P}_{n,i})\kappa_s, (\mathcal{I}-\mathcal{P}_{n,i})x\rangle_{\Delta_i}| \leq C_2 C_8 \big\|x^{(2r+1)}\big\|_\infty h^{2r+3}. \tag{4.5.7}$$

Combining the inequalities (4.5.5), (4.5.6) and (4.5.7), we conclude

$$|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s)| \leq (C_2)^2 C_8 \big\|x^{(2r+1)}\big\|_\infty h^{2r+3}. \tag{4.5.8}$$

Case II: Whenever $s = t_i$ for some $i = 0, 1, \ldots, n$. Since $\kappa_s \in \mathcal{C}^{2r+1}(\Delta_j)$ for each $j = 1, \ldots, n$ and given that $x \in \mathcal{C}^{2r+1}[0,1]$, we use inequality (4.3.3) to obtain

$$\begin{aligned} \max_{0\leq i\leq n} |\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(t_i)| &\leq \sum_{j=1}^{n} \|(\mathcal{I}-\mathcal{P}_{n,j})\kappa_s\|_{\Delta_j,\infty} \|(\mathcal{I}-\mathcal{P}_{n,j})x\|_{\Delta_j,\infty} h \\ &\leq (C_2)^2 C_8 \big\|x^{(2r+1)}\big\|_\infty h^{4r+2}. \end{aligned}$$

Finally, combining this estimate with (4.5.8), the desired result follow as

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_9 \big\|x^{(2r+1)}\big\|_\infty h^{2r+3},$$

where $C_9 = (C_2)^2 C_8$ is a constant independent of h.

To establish the other bound of the proposition (for $r = 0$), let us consider $x \in \mathcal{C}[0,1]$. Suppose $s = t_i$ for some index i. Then,

$$|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s)| \leq C_8 C_2 \|x\|_\infty h^{2r+1}. \tag{4.5.9}$$

Now, for $s \in (t_{i-1}, t_i)$, we express

$$\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \langle (\mathcal{I}-\mathcal{P}_{n,j})\kappa_s, x\rangle_{\Delta_j} + \langle (\mathcal{I}-\mathcal{P}_{n,i})(\kappa_s - \gamma_i), x\rangle_{\Delta_i},$$

which leads to the estimate

$$|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x(s)| \leq (1 + C_1 + C_2) C_8 \|x\|_\infty h^2.$$

By combining (4.5.9) with the above result, we arrive at

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)x\|_\infty \leq C_{10} \|x\|_\infty h^2,$$

where $C_{10} = (1 + C_1 + C_2) C_8$ is a constant. This completes the proof. □

Remark 4. *The higher-order estimate of order h^{2r+3} relies on the polynomial degree satisfying $r \geq 1$; for $r = 0$, the optimal bound reduces to order h^2.*

Lemma 4.5.1. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a Green's function type kernel. Let $\mathcal{P}_n$ be the orthogonal projection onto the approximating space $\mathcal{X}_n$ defined by* (4.3.1)*. Then*

$$\left\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+1}) & \text{for } r \geq 1, \\ O(h) & \text{for } r = 0, \end{cases}$$

and

$$\left\|\mathcal{K}(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^2) & \text{for } r = 0. \end{cases}$$

Proof. Note that

$$\left\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{ \|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1 \Big\}.$$

For $\psi \in \mathcal{R}(E)$, assume that $y = \mathcal{K}\psi$ and so $y \in \mathcal{R}(E)$. For $r \geq 1$, let us consider $y \in \mathcal{C}^{2r+1}[0,1]$ so that $\mathcal{K}y \in \mathcal{C}^{2r+1}[0,1]$ and therefore using (4.3.4), we have

$$\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty = \|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}y\|_\infty \leq C_2 \left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty h^{2r+1}.$$

From inequality (4.5.2), it follows that

$$\begin{aligned} \left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty &\leq C_8 \left(\|y\|_\infty + \|y'\|_\infty + \ldots + \left\|y^{(2r-1)}\right\|_\infty\right) \\ &= C_8 \left(\|\mathcal{K}\psi\|_\infty + \|(\mathcal{K}\psi)'\|_\infty + \ldots + \left\|(\mathcal{K}\psi)^{(2r-1)}\right\|_\infty\right). \end{aligned}$$

By utilizing inequalities (4.5.1) and (4.5.2), we conclude that the quantity $\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty$ is independent of h. Therefore, we obtain

$$\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty = O\big(h^{2r+1}\big),$$

which leads to

$$\left\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+1}\big).$$

In case $r = 0$, it is easy to see that

$$\left\|(\mathcal{K} - \mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O(h).$$

Now, consider the second bound of the lemma as

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)}\Big\{\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

Assume that $\psi \in \mathcal{C}^{2r+1}[0,1]$. Then $y \in \mathcal{C}^{2r+1}[0,1]$, and consequently $\mathcal{K}y \in \mathcal{C}^{2r+1}[0,1]$. Hence, by applying Proposition 39 for $r \geq 1$, we obtain

$$\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty = \|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}y\|_\infty \leq C_9\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty h^{2r+3}.$$

As discussed earlier, the quantity $\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty$ is independent of h and therefore

$$\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}\psi\|_\infty = O\big(h^{2r+3}\big)\,,$$

which implies that

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+3}\big)\,.$$

In the case $r = 0$, let $\psi \in \mathcal{C}[0,1]$ and set $y = \mathcal{K}\psi$. Then $\mathcal{K}y \in \mathcal{C}[0,1]$, and by Proposition 39 we obtain

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}y\|_\infty \leq C_{10}\|\mathcal{K}y\|_\infty h^2.$$

From estimate (4.5.1), we see that

$$\|\mathcal{K}y\|_\infty \leq C_8\|y\|_\infty = C_8\|\mathcal{K}\psi\|_\infty \leq (C_8)^2\|\psi\|_\infty.$$

Therefore, we get

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{P}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^2\big)\,,$$

which completes the proof.

□

Lemma 4.5.2. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a Green's function type kernel. Let $\mathcal{P}_n$ be the orthogonal projection onto the approximating space $\mathcal{X}_n$ defined by* (4.3.1). *Then*

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^3) & \text{for } r = 0, \end{cases}$$

and

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+5}) & \text{for } r \geq 1, \\ O(h^4) & \text{for } r = 0, \end{cases}$$

where $\mathcal{K}_n^M = \mathcal{P}_n\mathcal{K} + \mathcal{K}\mathcal{P}_n - \mathcal{P}_n\mathcal{K}\mathcal{P}_n$.

Proof. For any $\psi \in \mathcal{R}(E)$ we have $\mathcal{K}\psi \in \mathcal{R}(E)$. Then

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)}\Big\{\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

For $r \geq 1$, we use the bound

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \leq \|\mathcal{I}-\mathcal{P}_n\|\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty.$$

From (4.3.2), it follows that $\|\mathcal{I}-\mathcal{P}_n\| \leq 1 + C_1$. Let $\psi \in \mathcal{C}^{2r+1}[0,1]$, which implies $\mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$. Moreover, applying Proposition 39, we have

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \leq (1+C_1)C_9\left\|(\mathcal{K}\psi)^{(2r+1)}\right\|_\infty h^{2r+3}.$$

Following inequality (4.5.2), we observe that the bound $\left\|(\mathcal{K}\psi)^{(2r+1)}\right\|_\infty$ does not depend on h. Therefore,

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^{2r+3}\big)\,. \tag{4.5.10}$$

In case $r = 0$, let $\psi \in \mathcal{C}[0,1]$ and $y = \mathcal{K}\psi \in \mathcal{C}[0,1]$, then using (4.3.4), we have

$$\|(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty \leq C_2\left\|(\mathcal{K}(\mathcal{I}-\mathcal{P}_n)y)'\right\|_\infty h. \tag{4.5.11}$$

Now, define

$$\ell_s(t) = \ell(s,t) = \begin{cases} D^{(1,0)}\kappa_1(s,t), & s,t \in \Omega_1, \\ D^{(1,0)}\kappa_2(s,t), & s,t \in \Omega_2. \end{cases}$$

Consider

$$\begin{aligned}(\mathcal{K}(\mathcal{I}-\mathcal{P}_n)y)'(s) &= \int_0^1 \ell_s(t)(\mathcal{I}-\mathcal{P}_n)y(t)\,dt \\ &= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \ell_s(t)(\mathcal{I}-\mathcal{P}_{n,j})y(t)\,dt \\ &= \sum_{j=1}^n \langle(\mathcal{I}-\mathcal{P}_{n,j})\ell_s, (\mathcal{I}-\mathcal{P}_{n,j})y\rangle_{\Delta_j}.\end{aligned}$$

Let $s \in [t_{i-1}, t_i]$, for $i = 1, 2, \ldots, n$. Then, we can write

$$(\mathcal{K}(\mathcal{I}-\mathcal{P}_n)y)'(s) = \sum_{\substack{j=1\\ j\neq i}}^n \langle(\mathcal{I}-\mathcal{P}_{n,j})\ell_s, (\mathcal{I}-\mathcal{P}_{n,j})y\rangle_{\Delta_j} + \langle(\mathcal{I}-\mathcal{P}_{n,i})\ell_s, (\mathcal{I}-\mathcal{P}_{n,i})y\rangle_{\Delta_i}. \tag{4.5.12}$$

For $j \neq i$, we observe that both $\ell_s \in \mathcal{C}^{2r+1}(\Delta_j)$ and $x \in \mathcal{C}^{2r+1}(\Delta_j)$. Hence, utilizing the estimate (4.3.3), we obtain

$$\left|\sum_{\substack{j=1\\ j\neq i}}^n \langle(\mathcal{I}-\mathcal{P}_{n,j})\ell_s, (\mathcal{I}-\mathcal{P}_{n,j})y\rangle_{\Delta_j}\right| \leq (C_2)^2\|\ell_s\|_\infty\|y\|_\infty(n-1)h^3 \leq (C_2C_8)^2\|\psi\|_\infty(n-1)h^3, \tag{4.5.13}$$

where $\|\ell_s\|_\infty = C_8$ by definition. Also, whenever $j = i$, ℓ_s is continuous on $[t_{i-1}, t_i]$, for $i = 1, 2, \ldots, n$. Thus, following the proof similar to Proposition 39, we get

$$|\langle (\mathcal{I} - \mathcal{P}_{n,i})\ell_s, (\mathcal{I} - \mathcal{P}_{n,i})y\rangle_{\Delta_i}| = O\big(h^2\big)\,.$$

Applying the above estimate along with inequality (4.5.13) in (4.5.12), we have

$$\|(\mathcal{K}(\mathcal{I} - \mathcal{P}_n)y)'\|_\infty = O\big(h^2\big)$$

Using above estimate in (4.5.11), we obtain

$$\|(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^3\big)\,.$$

The above estimate and (4.5.10) establish the first bound of the lemma.
Now, consider the other bound of lemma as

$$\big\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\big\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

We can write it as follows

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty \leq \|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\|\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty.$$

For any $x \in \mathcal{C}[0,1]$, consider

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\| = \sup_{x \in \mathcal{C}[0,1]} \Big\{\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)x\|_\infty : \|x\|_\infty = 1\Big\}.$$

Using Proposition 39, we see that

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)x\|_\infty \leq C_{10}\|x\|_\infty h^2,$$

and so

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\| = O\big(h^2\big)\,.$$

For $r \geq 1$, let $\psi \in \mathcal{C}^{2r+1}[0,1]$, it follows that $\mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$ and using Proposition 39, we have

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^{2r+3}\big)\,.$$

Combining the above two estimates, we obtain

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^{2r+5}\big)\,. \tag{4.5.14}$$

In the case $r = 0$, let $\psi \in \mathcal{C}[0,1]$. Then $\mathcal{K}\psi \in \mathcal{C}[0,1]$, and by Proposition 39 we obtain

$$\|\mathcal{K}(\mathcal{I} - \mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^2\big)\,.$$

Therefore,

$$\|\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}(\mathcal{I}-\mathcal{P}_n)\mathcal{K}\psi\|_\infty = O\big(h^4\big)\,.$$

The above estimate along with (4.5.14) establish the desired second bound of the lemma, thereby completing the proof. □

By combining (4.3.7)-(4.3.9) with Lemma 4.5.1, we obtain the following convergence rates for the approximation of eigenvalues and spectral subspaces using the Galerkin and iterated Galerkin methods.

Theorem 40. *Let λ be a simple eigenvalue of $\mathcal{K}$, and let λ_n^G denote the corresponding Galerkin eigenvalue of $\mathcal{P}_n\mathcal{K}$. Let $\psi_n \in \mathcal{R}(E_n)$ be a Galerkin eigenfunction, and let ψ_n^S denote the associated iterated Galerkin eigenfunction. Then, for sufficiently large n,*

$$\|\psi_n - E\psi_n\|_\infty = \begin{cases} O(h^{2r+1}) & \text{for } r \geq 1, \\ O(h) & \text{for } r = 0, \end{cases}$$

$$\big\|\psi_n^S - E\psi_n^S\big\|_\infty = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^2) & \text{for } r = 0, \end{cases}$$

and

$$|\lambda - \lambda_n^G| = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^2) & \text{for } r = 0. \end{cases}$$

In a similar manner, by combining (4.3.10)-(4.3.12) with Lemma 4.5.2, we obtain the following convergence rates for the modified and iterated modified Galerkin methods.

Theorem 41. *Let λ be a simple eigenvalue of $\mathcal{K}$, and let λ_n^M denote the corresponding modified Galerkin eigenvalue of $\mathcal{K}_n^M$. Let $\psi_n^M \in \mathcal{R}(E_n^M)$ be a modified Galerkin eigenfunction, and let $\widetilde{\psi}_n^M$ denote the associated iterated modified Galerkin eigenfunction. Then, for sufficiently large n,*

$$\big\|\psi_n^M - E\psi_n^M\big\|_\infty = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^3) & \text{for } r = 0, \end{cases}$$

$$\Big\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\Big\|_\infty = \begin{cases} O(h^{2r+5}) & \text{for } r \geq 1, \\ O(h^4) & \text{for } r = 0, \end{cases}$$

and

$$|\lambda - \lambda_n^M| = \begin{cases} O(h^{2r+5}) & \text{for } r \geq 1, \\ O(h^4) & \text{for } r = 0. \end{cases}$$

Theorems 40 and 41 establish that the Iteration significantly improves the convergence rates of both the standard and modified Galerkin methods in the approximation of eigenfunctions. For $r \geq 1$, the modified method achieves higher-order convergence than the standard Galerkin method for both eigenfunctions and eigenvalues, and similar improvements are observed in the lowest-order case $r = 0$.

4.5.2 Interpolatory Projection Based Approximations

We first establish several preliminary results that will be used in the analysis of approximate eigenelements obtained using the interpolatory projection.

In particular, the following result is taken from Proposition 18 of Chapter 2.

Proposition 42. *For $r \geq 0$, if $x \in \mathcal{C}^{2r+2}[0,1]$, then*

$$\|\mathcal{K}(I - \mathcal{Q}_n)x\|_\infty \leq 2C_3\|x\|_{2r+2,\infty}h^{2r+2}.$$

Lemma 4.5.3. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a Green's function type kernel. Let $\mathcal{Q}_n$ be the interpolatory operator onto the approximating space $\mathcal{X}_n$ defined by* (4.3.5)*. Then for $r \geq 0$,*

$$\left\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+1}\big), \quad \left\|\mathcal{K}(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+2}\big).$$

Proof. Since the spectral subspace $\mathcal{R}(E)$ is invariant under $\mathcal{K}$, for any $\psi \in \mathcal{R}(E)$ we may define $y = \mathcal{K}\psi$, which implies that $y \in \mathcal{R}(E)$. Consider

$$\left\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

Further,

$$\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty = \|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}y\|_\infty.$$

Let $\psi \in \mathcal{C}^{2r+1}[0,1]$. Then $y \in \mathcal{C}^{2r+1}[0,1]$, and hence $\mathcal{K}y \in \mathcal{C}^{2r+1}[0,1]$. Therefore, by applying the estimate (4.3.6), we obtain

$$\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty \leq C_3\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty h^{2r+1}.$$

From inequality (4.5.2), we get

$$\begin{aligned}\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty &\leq C_8 \left(\|y\|_\infty + \|y'\|_\infty + \ldots + \left\|y^{(2r-1)}\right\|_\infty\right)\\ &= C_8 \left(\|\mathcal{K}\psi\|_\infty + \|(\mathcal{K}\psi)'\|_\infty + \ldots + \left\|(\mathcal{K}\psi)^{(2r-1)}\right\|_\infty\right).\end{aligned}$$

Following inequalities (4.5.1) and (4.5.2), we see that the bound $\left\|(\mathcal{K}y)^{(2r+1)}\right\|_\infty$ does not depend on h. Therefore,

$$\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty = O\big(h^{2r+1}\big).$$

Hence,

$$\left\|(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+1}\big).$$

Now, consider the second bound as

$$\left\|\mathcal{K}(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi \in \mathcal{R}(E)} \Big\{\|\mathcal{K}(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\Big\}.$$

Assume that $\psi \in \mathcal{C}^{2r+2}[0,1]$, which implies $y \in \mathcal{C}^{2r+2}[0,1]$ and so $\mathcal{K}y \in \mathcal{C}^{2r+2}[0,1]$. Using Proposition 42, we have

$$\|\mathcal{K}(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}\psi\|_\infty = \|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}y\|_\infty \leq 2C_3\|\mathcal{K}y\|_{2r+2,\infty}h^{2r+2}.$$

Note that

$$\|\mathcal{K}y\|_{2r+2,\infty} = \max_{0 \leq i \leq 2r+2} \left\|(\mathcal{K}y)^{(i)}\right\|_\infty.$$

From inequalities (4.5.1)-(4.5.3) discussed above, it follows that the bound $\left\|(\mathcal{K}y)^{(i)}\right\|_\infty$ is independent of h. Therefore, we finally obtain

$$\left\|\mathcal{K}(\mathcal{K} - \mathcal{Q}_n\mathcal{K})\mathcal{K}|_{\mathcal{R}(E)}\right\| = O\big(h^{2r+2}\big),$$

which completes the proof. □

Lemma 4.5.4. *Let $\mathcal{K}$ be the linear integral operator defined by* (4.1.2) *with a Green's function type kernel. Let $\mathcal{Q}_n$ be the interpolatory operator onto the approximating space $\mathcal{X}_n$ defined by* (4.3.5)*. Then*

$$\left\|(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+2}) & \text{for } r \geq 1,\\ O(h^3) & \text{for } r = 0,\end{cases}$$

and

$$\left\|\mathcal{K}(\mathcal{K} - \mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1,\\ O(h^4) & \text{for } r = 0,\end{cases}$$

where $\mathcal{K}_n^M = \mathcal{Q}_n\mathcal{K} + \mathcal{K}\mathcal{Q}_n - \mathcal{Q}_n\mathcal{K}\mathcal{Q}_n$.

Proof. Since $\mathcal{R}(E)$ is invariant under $\mathcal{K}$, we have $\mathcal{K}\psi \in \mathcal{R}(E)$ for all $\psi \in \mathcal{R}(E)$. Consider

$$\left\|(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)} \left\{\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\right\}.$$

For $r \geq 1$, we write

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq \|\mathcal{I}-\mathcal{Q}_n\|\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty.$$

Since $\{\mathcal{Q}_n\}$ is uniformly bounded, that is, $\|\mathcal{Q}_n\| = q < \infty$, and $\psi \in \mathcal{C}^{2r+2}[0,1]$, it follows that $\mathcal{K}\psi \in \mathcal{C}^{2r+2}[0,1]$. Hence, by Proposition 42, we obtain

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq 2C_3(1+q)\|\mathcal{K}\psi\|_{2r+2,\infty}h^{2r+2}.$$

Here

$$\|\mathcal{K}\psi\|_{2r+2,\infty} = \max_{0\leq i\leq 2r+2}\left\|(\mathcal{K}\psi)^{(i)}\right\|_\infty.$$

From inequalities (4.5.1)-(4.5.3), it follows that the bounds $\left\|(\mathcal{K}\psi)^{(i)}\right\|_\infty$ are independent of h for $i = 0,1,\ldots,2r+2$. Therefore,

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty = O\big(h^{2r+2}\big). \tag{4.5.15}$$

In case, $r = 0$, using (4.3.6), we have

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq C_3\left\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi)'\right\|_\infty h.$$

It follows from Chatelin-Lebbar [22, Theorem 15],

$$\left\|(\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi)'\right\|_\infty = O\big(h^2\big)$$

Therefore, from above two estimates, we have

$$\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty = O\big(h^3\big).$$

The above bound with (4.5.15) gives the the first estimate of the required results. Now, consider the other bound as

$$\left\|\mathcal{K}(\mathcal{K}-\mathcal{K}_n^M)\mathcal{K}|_{\mathcal{R}(E)}\right\| = \sup_{\psi\in\mathcal{R}(E)} \left\{\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty : \|\psi\|_\infty = 1\right\}.$$

For $r \geq 1$, we estimate

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq \|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\|\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty.$$

Since $\mathcal{K}$ is bounded linear operator, then

$$\begin{aligned}\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}x\|_\infty &\leq \|\mathcal{K}\|\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}x\|_\infty \\ &\leq \|\mathcal{K}\|C_3\|(\mathcal{K}x)''\|_\infty h^2 \\ &\leq C_3C_8\|\mathcal{K}\|\|x\|_\infty h^2.\end{aligned}$$

Taking the supremum over all $\|x\|_\infty = 1$, we conclude that

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\| = O\big(h^2\big)\,.$$

Let $\psi \in \mathcal{C}^{2r+2}[0,1] \subset \mathcal{C}^{2r+1}[0,1]$. Then $\mathcal{K}\psi \in \mathcal{C}^{2r+1}[0,1]$, and by applying estimate (4.3.6) together with the above bounds, we obtain

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty \leq \|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\|\|(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty = O\big(h^{2r+3}\big)\,.$$

In case, $r = 0$, let $\psi \in \mathcal{C}^2[0,1]$, which implies $\mathcal{K}\psi \in \mathcal{C}^2[0,1]$. Then, by Kulkarni-Nidhin [57, Proposition 3.7], we obtain

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)\mathcal{K}\psi\|_\infty = O\big(h^4\big)\,.$$

The above two bounds establish the second estimate of the required results, and complete the proof. □

By combining (4.3.7)-(4.3.9) with Lemma 4.5.3, we obtain the following orders of convergence for the approximation of eigenvalues and spectral subspaces using the collocation and iterated collocation methods.

Theorem 43. *Suppose λ is a simple eigenvalue of $\mathcal{K}$, and let λ_n^C be the eigenvalue of $\mathcal{Q}_n\mathcal{K}$ converging to λ. Let $\psi_n \in \mathcal{R}(E_n)$ be a collocation eigenfunction, and let ψ_n^S denote the corresponding iterated collocation eigenfunction. Then, for all n large enough*

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \big\|\psi_n^S - E\psi_n^S\big\|_\infty = O\big(h^{2r+2}\big)\,,$$

and

$$|\lambda - \lambda_n^C| = O\big(h^{2r+2}\big)\,.$$

Similarly, by combining (4.3.10)-(4.3.12) with Lemma 4.5.4, we obtain the following orders of convergence for the modified and iterated modified collocation methods.

Theorem 44. *Suppose λ is a simple eigenvalue of $\mathcal{K}$, and let λ_n^M be the eigenvalue of $\mathcal{K}_n^M$ converging to λ. Let $\psi_n^M \in \mathcal{R}(E_n^M)$ be a modified collocation eigenfunction, and let $\widetilde{\psi}_n^M$ denote the corresponding iterated modified collocation eigenfunction. Then, for all n large enough*

$$\left\|\psi_n^M - E\psi_n^M\right\|_\infty = \begin{cases} O(h^{2r+2}) & \text{for } r \geq 1, \\ O(h^3) & \text{for } r = 0, \end{cases}$$

$$\left\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\right\|_\infty = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^4) & \text{for } r = 0, \end{cases}$$

and

$$|\lambda - \lambda_n^M| = \begin{cases} O(h^{2r+3}) & \text{for } r \geq 1, \\ O(h^4) & \text{for } r = 0. \end{cases}$$

The above results show that a single iteration improves the convergence rate of both the standard and modified collocation methods in the approximation of eigenfunctions. For $r \geq 1$, the modified method achieves higher-order convergence than the standard collocation method for both eigenvalues and spectral subspaces. Even in the lowest-order case $r = 0$, the modified collocation variants provide substantial improvements.

4.6 Numerical Results

In this section, we present numerical experiments to illustrate and validate the theoretical convergence results established in the previous sections. Numerical results are reported for operators with both smooth kernels and Green's function-type kernels.

Let $\mathcal{X}_n$ denote the space of piecewise constant functions associated with the uniform partition of the interval $[0, 1]$ given by

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n.$$

Throughout the numerical experiments, the experimental order of convergence is computed using the standard formula

$$\delta = \frac{\log(e_n/e_{2n})}{\log 2},$$

where e_n denotes the numerical error obtained using n subintervals.

Orthogonal Projection. Let $\mathcal{P}_n : \mathcal{L}^2[0, 1] \to \mathcal{X}_n$ be the orthogonal projection onto the subspace of piecewise constants functions. We denote by λ_n^G and λ_n^{MG} the eigenvalue approximations obtained from the Galerkin and modified Galerkin methods, respectively. The corresponding convergence rates are denoted by $\hat{\delta}_G$ and $\hat{\delta}_{MG}$.

Interpolatory Projection. Let $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denote the interpolatory projection defined by

$$(\mathcal{Q}_n x)(\tau_j) = x(\tau_j), \quad j = 1, 2, \ldots, n,$$

where the collocation points $\{\tau_j\}$ are chosen as the midpoints of the subintervals:

$$\tau_j = \frac{2j-1}{2n}, \quad j = 1, 2, \ldots, n.$$

We denote by λ_n^C and λ_n^{MC} the eigenvalue approximations obtained using the collocation and modified collocation methods, respectively, with $\hat{\delta}_C$ and $\hat{\delta}_{MC}$ representing their respective orders of convergence.

Smooth Kernel

We first consider the integral operator $\mathcal{K}$ defined by the smooth kernel

$$\kappa(s,t) = e^{st}, \quad s, t \in [0,1],$$

with the operator action defined as:

$$(\mathcal{K}x)(s) = \int_0^1 e^{st} x(t)\, dt, \quad s \in [0,1].$$

The eigenvalues of $\mathcal{K}$ are known to be simple; specifically, the largest eigenvalue is $\lambda = 1.35303016 47457353$. Numerical comparisons between standard and modified versions of both Galerkin and Collocation methods are provided in Tables 4.6.1 and 4.6.2.

Table 4.6.1: Eigenvalue errors in variants of Galerkin methods for the smooth kernel.

n	$\lvert\lambda - \lambda_n^G\rvert$	$\hat{\delta}_G$	$\lvert\lambda - \lambda_n^{MG}\rvert$	$\hat{\delta}_{MG}$
2	9.69×10^{-3}	–	2.68×10^{-4}	–
4	2.49×10^{-3}	1.96	1.76×10^{-5}	3.93
8	6.26×10^{-4}	1.99	1.12×10^{-6}	3.98
16	1.57×10^{-4}	2.00	7.01×10^{-8}	4.00
32	3.92×10^{-5}	2.00	4.38×10^{-9}	4.00
64	9.81×10^{-6}	2.00	2.74×10^{-10}	4.00

The results demonstrate that the standard projection method achieves second-order convergence, while the modified projection method attains fourth-order superconvergence.

Table 4.6.2: Eigenvalue errors in variants of collocation methods for the smooth kernel.

n	$\lvert\lambda - \lambda_n^C\rvert$	$\hat{\delta}_C$	$\lvert\lambda - \lambda_n^{MC}\rvert$	$\hat{\delta}_{MC}$
2	2.08×10^{-2}	–	6.25×10^{-4}	–
4	5.47×10^{-3}	1.93	4.20×10^{-5}	3.90
8	1.39×10^{-3}	1.98	2.67×10^{-6}	3.97
16	3.48×10^{-4}	2.00	1.68×10^{-7}	3.99
32	8.70×10^{-5}	2.00	1.05×10^{-8}	4.00
64	2.18×10^{-5}	2.00	6.56×10^{-10}	4.00

The orders of convergence for the correponding eigenfunctions can be found in numerical results section of Kulkarni [49, 52] and Long et al. [60]. The results confirm the modified method's superior accuracy, matching theoretical predictions exactly.

Green's Kernel

We now consider the compact integral operator $\mathcal{K}$ defined by

$$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt,$$

with the Green's function-type kernel κ is given by

$$\kappa(s,t) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

The kernel $\kappa \in \mathcal{C}(\Omega)$, and it is well known that all eigenvalues of $\mathcal{K}$ are simple. The largest eigenvalue is given explicitly by $\lambda = \dfrac{1}{\pi^2}$, with the corresponding eigenfunction $\varphi(s) = \sin(\pi s)$.

Galerkin methods and variants: Using the orthogonal projection $\mathcal{P}_n : \mathcal{L}^2[0,1] \to \mathcal{X}_n$, Table 4.6.3 shows the eigenvalue errors. The errors in the approximation of the largest eigenvalue, alongside their corresponding convergence rates, are summarized in Table 4.6.3. The results demonstrate that the standard Galerkin method yields second-order convergence, whereas the modified Galerkin method exhibits a significantly enhanced fourth-order rate.

The accuracy of the corresponding eigenfunction approximations is examined next. Table 4.6.4 presents the errors measured in the supremum norm, where the error is computed in terms of the gap between the exact spectral subspace and its numerical approximation. The results demonstrate that the Galerkin method exhibits first-order convergence of eigenfunctions. The iterated Galerkin and modified Galerkin methods improve the rate to second and third order, respectively. In particular, the iterated modified Galerkin method achieves fourth-order convergence. These results numerically validate the super-convergence effects predicted by the theoretical analysis.

Table 4.6.3: Eigenvalue errors in variants of Galerkin methods for Green's kernel.

n	$\|\lambda - \lambda_n^G\|$	$\hat{\delta}_G$	$\|\lambda - \lambda_n^M\|$	$\hat{\delta}_{MG}$
2	1.80×10^{-2}		3.75×10^{-3}	
4	5.04×10^{-3}	1.83	2.95×10^{-4}	3.67
8	1.29×10^{-3}	1.96	1.96×10^{-5}	3.91
16	3.25×10^{-4}	1.99	1.25×10^{-6}	3.97
32	8.13×10^{-5}	2.00	8.06×10^{-8}	3.96
64	2.03×10^{-5}	2.00	4.91×10^{-9}	4.04

Table 4.6.4: Eigenfunction errors in variants of Galerkin methods for Green's kernel.

n	$\\|\psi_n^G - E\psi_n^G\\|_\infty$	δ_G	$\\|\psi_n^S - E\psi_n^S\\|_\infty$	δ_{IG}	$\\|\psi_n^M - E\psi_n^M\\|_\infty$	δ_{MG}	$\\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\\|_\infty$	δ_{IMG}
2	6.23×10^{-1}		2.34×10^{-2}		1.11×10^{-2}		3.02×10^{-4}	
4	3.71×10^{-1}	0.75	5.81×10^{-3}	2.01	1.87×10^{-3}	2.56	1.83×10^{-5}	4.04
8	1.94×10^{-1}	0.94	1.45×10^{-3}	2.00	2.50×10^{-4}	2.90	9.57×10^{-7}	4.26
16	9.78×10^{-2}	0.98	3.62×10^{-4}	2.00	3.18×10^{-5}	2.98	6.03×10^{-8}	3.99
32	4.90×10^{-2}	1.00	9.06×10^{-5}	2.00	3.99×10^{-6}	2.99	3.78×10^{-9}	4.00
64	2.45×10^{-2}	1.00	2.26×10^{-5}	2.00	4.99×10^{-7}	3.00	2.36×10^{-10}	4.00

Collocation methods and variants: Similar behaviour is observed for interpolatory projections at midpoints onto the approximating spaces of piecewise constant functions. The corresponding eigenvalue errors and experimental convergence rates are reported in Table 4.6.5. The numerical results confirm second-order convergence for the collocation method and fourth-order convergence for the modified collocation method.

Finally, Table 4.6.6 reports the errors for eigenfunction approximations obtained using collocation-based methods. The results demonstrate that the modified and iterated modified collocation methods significantly improve the convergence rate, as iterated modified collocation eigenfunction achieves the highest fourth-order accuracy.

Overall, the numerical experiments strongly support the theoretical analysis and highlight the superior performance of the modified projection-based methods across different kernel types.

Table 4.6.5: Eigenvalue errors in variants of collocation methods for Green's kernel.

n	$\lvert\lambda - \lambda_n^C\rvert$	$\hat{\delta}_C$	$\lvert\lambda - \lambda_n^M\rvert$	$\hat{\delta}_{MC}$
4	2.44×10^{-3}		6.60×10^{-4}	
8	6.41×10^{-4}	1.93	4.07×10^{-5}	4.02
16	1.62×10^{-4}	1.98	2.54×10^{-6}	4.00
32	4.07×10^{-5}	2.00	1.59×10^{-7}	4.00
64	1.02×10^{-5}	2.00	9.93×10^{-9}	4.00
128	2.54×10^{-6}	2.00	6.21×10^{-10}	4.00

Table 4.6.6: Eigenfunction errors in variants of collocation methods for Green's kernel.

n	$\\|\psi_n^C - E\psi_n^C\\|_\infty$	δ_C	$\\|\psi_n^S - E\psi_n^S\\|_\infty$	δ_{IC}	$\\|\psi_n^M - E\psi_n^M\\|_\infty$	δ_{MC}	$\\|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\\|_\infty$	δ_{IMC}
4	4.14×10^{-1}		7.59×10^{-2}		1.07×10^{-3}		9.31×10^{-5}	
8	1.99×10^{-1}	1.06	1.91×10^{-2}	1.99	1.37×10^{-4}	2.96	5.70×10^{-6}	4.03
16	9.85×10^{-2}	1.01	4.81×10^{-3}	1.99	1.73×10^{-5}	2.99	3.49×10^{-7}	4.03
32	4.91×10^{-2}	1.00	1.18×10^{-3}	2.02	2.16×10^{-6}	3.00	1.54×10^{-8}	4.50
64	2.45×10^{-2}	1.00	2.97×10^{-4}	2.00	2.70×10^{-7}	3.00	1.51×10^{-9}	3.36
128	1.23×10^{-2}	1.00	7.41×10^{-5}	2.00	3.37×10^{-8}	3.00	9.43×10^{-11}	4.00

4.7 Conclusion

In this chapter, we have investigated the convergence behaviour of projection-based approximations for eigenvalue problems associated with compact integral operators having Green's kernels. Both classical and modified projection methods were analyzed in a unified framework. We established explicit convergence rates for the approximate eigenvalues and the corresponding spectral subspaces in the supremum-norm. Our analysis shows that the modified (iterated) projection method produces higher-order convergence of eigenvalues compared to the classical projection method. In particular, the superconvergence phenomenon arising from iteration significantly improves the approximation of spectral subspaces. This enhancement persists even when the underlying Green's kernel lacks smoothness along the diagonal, thereby extending the scope of earlier results that were primarily restricted to smooth kernels. The theoretical findings are supported by numerical experiments, which confirm that the iterated method achieves higher accuracy with a computational effort. Consequently, the modified projection approach provides an efficient and reliable framework for eigenvalue computations involving Green's kernels.

Chapter 5

Summary and Future Scope

Let the uniform partition of $[0,1]$ into n equal parts,

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad t_j = \frac{j}{n}, \quad j = 0, 1, \ldots, n,$$

with subintervals $\Delta_j = [\,t_{j-1},\, t_j\,], \quad j = 1, 2, \ldots, n$. For a non-negative integer r, define the finite-dimensional approximating space

$$\mathcal{X}_n = \left\{ x \in \mathcal{L}^\infty[0,1] : x|_{\Delta_j} \text{ is a polynomial of degree} \leq 2r,\ j = 1, 2, \ldots, n \right\}.$$

Thus, $\mathcal{X}_n$ denotes the space of piecewise polynomials of even degree $\leq 2r$, with $\dim \mathcal{X}_n = n(2r+1)$ and $\mathcal{X}_n \subset \mathcal{L}^\infty[0,1]$. We consider the orthogonal projection $\mathcal{P}_n$ and the interpolatory projection $\mathcal{Q}_n$ mapping into $\mathcal{X}_n$. For $\mathcal{Q}_n$, on each subinterval $\Delta_j = [t_{j-1}, t_j]$ we choose $2r+1$ equidistant interpolation points given by

$$\tau_j^i = t_{j-1} + \frac{ih}{2r}, \quad i = 0, 1, \ldots, 2r,\ r \geq 1, \quad \text{and} \quad \tau_j = \frac{t_{j-1} + t_j}{2},\ r = 0.$$

These collocation points are equidistant on each subinterval and are not required to be Gauss points.

1. Let $\mathcal{X} = \mathcal{C}[0,1]$, and consider the Fredholm integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ defined by

   $$(\mathcal{K}x)(s) = \int_0^1 \kappa(s,t)\, x(t)\, dt, \quad s \in [0,1],\ x \in \mathcal{X},$$

   where $\kappa : \Omega \to \mathbb{R}$ is a given continuous kernel function and $\Omega = [0,1] \times [0,1]$. We are interested in the numerical approximation of the Fredholm integral equation of the second kind

   $$x - \mathcal{K}x = f, \quad f \in \mathcal{X},$$

which is assumed to admit a unique solution φ. Let $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denote the interpolatory projection operator. The following projection-based approximation methods are considered.

Collocation method.

$$\varphi_n^C - \mathcal{Q}_n \mathcal{K} \varphi_n^C = \mathcal{Q}_n f.$$

Iterated collocation method.

$$\varphi_n^S = \mathcal{K} \varphi_n^C + f.$$

The modified collocation operator $\mathcal{K}_n^M$ is defined by

$$\mathcal{K}_n^M = \mathcal{Q}_n \mathcal{K} + \mathcal{K} \mathcal{Q}_n - \mathcal{Q}_n \mathcal{K} \mathcal{Q}_n.$$

Modified collocation method.

$$\varphi_n^M - \mathcal{K}_n^M \varphi_n^M = f.$$

Iterated modified collocation method.

$$\widetilde{\varphi}_n^M = \mathcal{K} \varphi_n^M + f.$$

Smooth kernel case. Assume that the kernel κ of the Fredholm integral operator is smooth, with $\kappa \in \mathcal{C}^{2r+2}(\Omega)$, and that $f \in \mathcal{C}^{2r+2}[0,1]$. Under these assumptions, the collocation solution and its variants satisfy the following convergence estimates:

$$\begin{aligned} \left\|\varphi - \varphi_n^C\right\|_\infty &= O\big(h^{2r+1}\big), & \left\|\varphi - \varphi_n^S\right\|_\infty &= O\big(h^{2r+2}\big), \\ \left\|\varphi - \varphi_n^M\right\|_\infty &= O\big(h^{4r+3}\big), & \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty &= O\big(h^{4r+4}\big). \end{aligned}$$

Green's function kernel case. Assume that the kernel κ of the Fredholm integral operator is of Green's function type, satisfying $\kappa \in \mathcal{C}(\Omega)$, $\kappa \in \mathcal{C}^\alpha(\Omega_1) \cap \mathcal{C}^\alpha(\Omega_2)$, indicates piecewise $\mathcal{C}^\alpha$ regularity with possible jumps across $s = t$ for an integer $\alpha \geq 0$, where $\Omega_1 = \{0 \leq t \leq s \leq 1\}$, $\Omega_2 = \{0 \leq s \leq t \leq 1\}$. Assume further that the right-hand side $f \in \mathcal{C}^{2r+2}[0,1]$. Under these assumptions, the following estimates hold:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big), \qquad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+2}\big),$$

$$\left\|\varphi - \varphi_n^M\right\|_\infty = \begin{cases} O(h^3), & r = 0, \\ O(h^{2r+2}), & r \geq 1, \end{cases} \qquad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = \begin{cases} O(h^4), & r = 0, \\ O(h^{2r+3}), & r \geq 1. \end{cases}$$

The results for Green's kernels are reported in Rakshit-Shukla-Rane [72] and incorporated in Chapter 2 of this thesis.

2. Let $\mathcal{X} = \mathcal{L}^{\infty}[0,1]$, and consider the Urysohn integral operator $\mathcal{K} : \mathcal{X} \to \mathcal{C}[0,1]$ defined by

$$\mathcal{K}(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1],\ x \in \mathcal{X},$$

where $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We are interested in the numerical approximation of the nonlinear integral equation

$$x - \mathcal{K}(x) = f, \quad f \in \mathcal{X}.$$

We assume that the above equation admits a unique solution φ. Let $\mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denote the interpolatory projection operator. The following projection-based approximation methods are considered.

Collocation method.

$$\varphi_n^C - \mathcal{Q}_n\mathcal{K}(\varphi_n^C) = \mathcal{Q}_n f.$$

Iterated collocation method.

$$\varphi_n^S = \mathcal{K}(\varphi_n^C) + f.$$

The modified collocation operator $\mathcal{K}_n^M$ is defined by

$$\mathcal{K}_n^M(x) = \mathcal{Q}_n\mathcal{K}(x) + \mathcal{K}(\mathcal{Q}_n x) - \mathcal{Q}_n\mathcal{K}(\mathcal{Q}_n x).$$

Modified collocation method.

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f.$$

Iterated modified collocation method.

$$\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f.$$

Smooth kernel case. Assume that the smooth kernel κ of the Urysohn integral operator satisfies κ, $\dfrac{\partial\kappa}{\partial u} \in \mathcal{C}^{2r+1}(\Omega \times \mathbb{R})$, $\dfrac{\partial^2\kappa}{\partial u^2} \in \mathcal{C}^{2r+2}(\Omega \times \mathbb{R})$, and that $f \in \mathcal{C}^{2r+2}[0,1]$. Under these smoothness assumptions, the collocation solution and their variants satisfy the following error estimates:

$$\left\|\varphi - \varphi_n^C\right\|_\infty = O\big(h^{2r+1}\big), \qquad \left\|\varphi - \varphi_n^S\right\|_\infty = O\big(h^{2r+2}\big),$$

$$\left\|\varphi - \varphi_n^M\right\|_\infty = O\big(h^{4r+3}\big), \qquad \left\|\varphi - \widetilde{\varphi}_n^M\right\|_\infty = O\big(h^{4r+4}\big).$$

Green's function kernel case. Assume that the Green's kernel κ of the Urysohn integral operator satisfies

$$\kappa, \ \frac{\partial^i \kappa}{\partial u^i} \in \mathcal{C}(\Omega \times \mathbb{R}), \quad i = 1, 2, 3, 4,$$

and that

$$\frac{\partial \kappa}{\partial u}, \frac{\partial^2 \kappa}{\partial u^2} \in \mathcal{C}^{\alpha}(\Omega_1 \times \mathbb{R}) \cap \mathcal{C}^{\alpha}(\Omega_2 \times \mathbb{R}), \quad \Omega = \Omega_1 \cup \Omega_2,$$

with possible jump discontinuities across the diagonal $s = t$. Moreover, assume that $f \in \mathcal{C}^{2r+2}[0, 1]$. Under these smoothness assumptions, the methods of approximations satisfy the error estimates

$$\left\| \varphi - \varphi_n^C \right\|_\infty = O\big(h^{2r+1}\big), \qquad \left\| \varphi - \varphi_n^S \right\|_\infty = O\big(h^{2r+2}\big),$$

$$\left\| \varphi - \varphi_n^M \right\|_\infty = \begin{cases} O(h^3), & r = 0, \\ O(h^{2r+2}), & r \geq 1, \end{cases} \qquad \left\| \varphi - \widetilde{\varphi}_n^M \right\|_\infty = \begin{cases} O(h^4), & r = 0, \\ O(h^{2r+3}), & r \geq 1. \end{cases}$$

The above results have been published in Shukla-Rakshit [81] and are presented in detail in Chapter 3.

3. In addition to the integral equation of the second kind, we consider the eigenvalue problem associated with a compact Fredholm integral operator $\mathcal{K}$ defined on $\mathcal{X} = \mathcal{L}^2[0, 1]$ or $\mathcal{C}[0, 1]$. Let κ be a real-valued continuous function on $\Omega = [0, 1] \times [0, 1]$, and let $\mathcal{K} : \mathcal{X} \to \mathcal{X}$ denote the corresponding compact linear integral operator. We seek $\lambda \in \mathbb{C} \setminus \{0\}$ and $0 \neq \psi \in \mathcal{X}$ such that

$$\mathcal{K}\psi = \lambda\psi.$$

Our objective is to construct approximate eigenpairs (λ_n, ψ_n) converging to (λ, ψ). Let $\pi_n : \mathcal{X} \to \mathcal{X}_n$ be a projection (either orthogonal or interpolatory) operator. The classical projection method replaces $\mathcal{K}$ by $\pi_n\mathcal{K}$, leading to

$$\pi_n \mathcal{K} \psi_n = \lambda_n \psi_n, \quad \psi_n \in \mathcal{X}_n, \ \|\psi_n\|_\infty = 1.$$

A one-step refinement defines the iterated eigenvector

$$\psi_n^S = \frac{1}{\lambda_n} \mathcal{K} \psi_n.$$

In the modified projection method, the operator $\mathcal{K}$ is replaced by

$$\mathcal{K}_n^M = \pi_n \mathcal{K} + \mathcal{K} \pi_n - \pi_n \mathcal{K} \pi_n,$$

leading to the modified eigenvalue problem

$$\mathcal{K}_n^M \psi_n^M = \lambda_n^M \psi_n^M, \quad \psi_n^M \in \mathcal{X}, \ \left\|\psi_n^M\right\|_\infty = 1,$$

with the associated iterated modified eigenvector

$$\widetilde{\psi}_n^M = \frac{1}{\lambda_n^M} \mathcal{K} \psi_n^M.$$

Let λ be an isolated eigenvalue of $\mathcal{K}$ with finite algebraic multiplicity m, and let E denote the corresponding spectral projection. Let E_n and E_n^M be the spectral projections of $\pi_n \mathcal{K}$ and $\mathcal{K}_n^M$, respectively, associated with eigenvalues enclosed by a simple closed contour Γ containing λ. Denote by $\hat{\lambda}_n$ and $\hat{\lambda}_n^M$ the arithmetic means of the m eigenvalues of $\pi_n \mathcal{K}$ and $\mathcal{K}_n^M$ inside Γ.

The convergence results for the approximation of a simple eigenvalue λ and the associated spectral subspaces using both orthogonal and interpolatory projection methods are summarized below.

Orthogonal Projection. Let $\pi_n = \mathcal{P}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denote the orthogonal projection operator.

Smooth kernel case. Assume that the kernel κ of the compact Fredholm integral operator $\mathcal{K}$ is smooth, with $\kappa \in \mathcal{C}^{2r+2}(\Omega)$. Then the approximations of eigenvalues and associated eigenspaces, measured in the gap metric, satisfy

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \|\psi_n^S - E\psi_n^S\|_\infty = O\big(h^{4r+2}\big)\,, \quad |\lambda - \lambda_n^G| = O\big(h^{4r+2}\big)\,,$$

$$\|\psi_n^M - E\psi_n^M\|_\infty = O\big(h^{6r+3}\big)\,, \quad \|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\|_\infty = O\big(h^{8r+4}\big)\,, \quad |\lambda - \lambda_n^M| = O\big(h^{8r+4}\big)\,.$$

Green's function kernel case. Assume that κ is of Green's function type, with piecewise $\mathcal{C}^\alpha$ regularity on Ω_1 and Ω_2 for some integer $\alpha \geq 0$, and with possible jump discontinuities across the diagonal $s = t$. Then the following convergence estimates hold:

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \|\psi_n^S - E\psi_n^S\|_\infty = O\big(h^{2r+3}\big)\,, \quad |\lambda - \lambda_n^G| = O\big(h^{2r+3}\big)\,,$$

$$\|\psi_n^M - E\psi_n^M\|_\infty = \begin{cases} O(h^3)\,, & r = 0, \\ O(h^{2r+3})\,, & r \geq 1, \end{cases} \qquad \|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\|_\infty = \begin{cases} O(h^4)\,, & r = 0, \\ O(h^{2r+5})\,, & r \geq 1, \end{cases}$$

$$|\lambda - \lambda_n^M| = \begin{cases} O(h^4)\,, & r = 0, \\ O(h^{2r+5})\,, & r \geq 1. \end{cases}$$

Interpolatory Projection. Let $\pi_n = \mathcal{Q}_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ denote the interpolatory projection operator.

Smooth kernel case. Suppose that the kernel κ of the compact Fredholm integral operator $\mathcal{K}$ is smooth, with $\kappa \in \mathcal{C}^{2r+2}(\Omega)$. Then the approximations of eigenvalues and the associated eigenspaces, measured in the gap metric, satisfy the following convergence rates:

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \|\psi_n^S - E\psi_n^S\|_\infty = O\big(h^{2r+2}\big)\,, \quad |\lambda - \lambda_n^C| = O\big(h^{2r+2}\big)\,,$$

$$\|\psi_n^M - E\psi_n^M\|_\infty = O\big(h^{4r+3}\big)\,, \quad \|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\|_\infty = O\big(h^{4r+4}\big)\,, \quad |\lambda - \lambda_n^M| = O\big(h^{4r+4}\big)\,.$$

Green's function kernel case. Assume that κ is of Green's function type, with piecewise $\mathcal{C}^\alpha$ regularity on Ω_1 and Ω_2 for an integer $\alpha \geq 0$, and with possible jump discontinuities across the diagonal $s = t$. Then the approximations of the eigenelements satisfy

$$\|\psi_n - E\psi_n\|_\infty = O\big(h^{2r+1}\big)\,, \quad \|\psi_n^S - E\psi_n^S\|_\infty = O\big(h^{2r+2}\big)\,, \quad |\lambda - \lambda_n^C| = O\big(h^{2r+2}\big)\,,$$

$$\|\psi_n^M - E\psi_n^M\|_\infty = \begin{cases} O(h^3)\,, & r = 0, \\ O(h^{2r+2})\,, & r \geq 1, \end{cases} \qquad \|\widetilde{\psi}_n^M - E\widetilde{\psi}_n^M\|_\infty = \begin{cases} O(h^4)\,, & r = 0, \\ O(h^{2r+3})\,, & r \geq 1, \end{cases}$$

$$|\lambda - \lambda_n^M| = \begin{cases} O(h^4)\,, & r = 0, \\ O(h^{2r+3})\,, & r \geq 1. \end{cases}$$

These results follow from the collectively compact convergence of the finite-rank projection operators and the spectral approximation theory for compact operators presented in Chapter 4. The corresponding results for Green's kernels are reported in Shukla-Rakshit-Rane [82].

Limitations of the Present Work

Although the present thesis establishes higher–order convergence results for collocation, iterated, and modified projection methods for linear Fredholm, nonlinear Urysohn, and associated eigenvalue problems, the analysis is confined to compact integral operators defined on one-dimensional domains under uniform partitions. The theoretical estimates rely on suitable smoothness or piecewise smoothness assumptions on the kernels (including Green's function type kernels with controlled jump discontinuities across the diagonal), and therefore operators with stronger singularities, weakly singular kernels of non-integrable type, or problems posed on irregular geometries are not covered. The approximation spaces considered consist of piecewise polynomials of even degree on uniform meshes; adaptive discretizations, non-uniform partitions, spline or wavelet-based approximation spaces, and higher-dimensional domains have not been investigated. In the eigenvalue analysis, the projection-based convergence results are derived under the assumption of isolated simple eigenvalues; the treatment of multiple or clustered eigenvalues are not included here. Future research may focus on extending the proposed projection-based framework to weakly singular kernels, adaptive discretizations, higher-dimensional domains, and multiple eigenvalue problems.

Future Scope of Research

The projection-based methods developed in this thesis establish a rigorous continuous framework for high-order and superconvergent approximations of Fredholm and Urysohn integral equations and the associated eigenvalue problems. Since practical computation requires the use of numerical quadrature, a natural and important direction for future research is the development of fully *discrete projection-based methods.* Discrete projection methods are particularly attractive because they can preserve the convergence orders predicted by the continuous theory while significantly reducing computational complexity. See Kulkarni-Rakshit [54, 55]. Unlike classical Nyström methods, which often require very fine quadrature meshes and large matrix systems, discrete iterated modified projection methods have the potential to achieve comparable or higher accuracy using substantially smaller systems. The following research directions are of particular interest:

- Development and analysis of fully *discrete modified projection and iterated modified projection methods*, incorporating suitable quadrature rules and preserving the convergence and superconvergence orders of the continuous theory for integral equations and associated eigenvalue problems, with special emphasis on *non-smooth kernels of Green's function type* using interpolatory projections.

- Extension of projection-based methods to *singular and weakly singular integral equa-*

tions. In particular, motivated by the one-dimensional Fredholm and Urysohn integral equations with smooth and Green's function type kernels studied in this thesis, we plan to investigate *linear weakly singular integral equations* and to develop *two-grid methods based on discrete modified projection operators*, combined with product integration techniques and rigorous error analysis.

- Construction of *asymptotic series expansions for iterated projection solutions* of operator equations and eigenvalue problems to obtain further acceleration of convergence.

- Application of projection-based methods to *boundary integral equations* arising from Dirichlet problems for Laplace's equation on polygonal domains and to integral equations defined on surfaces.

Papers based on Thesis work

1. G. Rakshit, S. K. Shukla, and A. S. Rane, A note on improvement by iteration for the approximate solutions of second kind Fredholm integral equations with Green's kernels, *The Journal of Analysis*, vol. 33, pp. 1599–1621, 2025.
   DOI: 10.1007/s41478-025-00882-0.

2. S. K. Shukla and G. Rakshit, Acceleration of convergence in approximate solutions of Urysohn integral equations with Green's kernels, *Mathematics and Computers in Simulation*, vol. 240, pp. 681–697, 2026.
   DOI: 10.1016/j.matcom.2025.07.044.

3. S. K. Shukla, G. Rakshit, and A. S. Rane, Projection-based approximations for eigenvalue problems of Fredholm integral operators with Green's kernels, *Numerical Algorithms*, 2026.
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