Extended Abstract

Analysis on the Approximate Solutions of Integral Equations and Eigenvalue Problems

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Abstract

Integral equations arise naturally in a wide range of scientific and engineering applications, often as reformulations of boundary value problems in physics, fluid dynamics, elasticity, and potential theory. This thesis focuses on the numerical analysis of Fredholm integral equations of the second kind - both linear and nonlinear (Urysohn) - as well as the associated eigenvalue problems for compact linear integral operators. In many practical problems, finding an exact solution is very difficult or even impossible. This is usually because the kernel function or the data may be complicated. For this reason, we employ finite-rank approximation techniques based on projection methods to obtain accurate and computationally feasible numerical solutions.

Our research focuses on the numerical solution of Fredholm integral equations of the second kind, where $\mathcal{K}$ may be linear or nonlinear (Urysohn) integral operator, and on the associated eigenvalue problem for linear integral operators. Since exact solutions are rarely available, the operator $\mathcal{K}$ is approximated by finite-rank operators $\mathcal{K}_n$, reducing both the integral equation and the eigenvalue problem to finite-dimensional linear or nonlinear systems that can be solved computationally. Two main approaches are used to construct $\mathcal{K}_n$: quadrature-based Nyström operators, in which the integral is replaced by a convergent quadrature formula, and projection-based methods, which rely on a sequence of finite-rank projection operators converging pointwise to the identity operator, such as the Galerkin and collocation methods, including their iterated variants. We also study the *modified projection method*, which achieves significantly improved convergence without increasing computational complexity. Convergence analysis and error estimates are developed using tools from Functional Analysis.

This thesis develops an analytical framework for projection-based numerical methods for Fredholm and Urysohn integral equations and for the associated eigenvalue problems of compact linear integral operators. For both smooth and Green's function-type kernels, we establish convergence results for collocation, iterated collocation, modified collocation, and iterated modified collocation methods based on interpolatory projections at equidistant points, and for eigenvalue problems we additionally incorporate orthogonal projections to obtain accurate spectral approximations. A central contribution is the rigorous demonstration that the modified collocation methods exhibit superior convergence when compared with classical collocation schemes, for both operator equations and eigenvalue problems, together with improved convergence rates for eigenvalues and their corresponding spectral subspaces. Furthermore, we show that such an appropriate choice of interpolation points yields accelerated convergence for the iterated forms of both the standard and modified collocation methods. We also employ discrete projection-based methods for the operator equation and the eigenvalue problem with smooth kernels, identifying conditions that preserve the continuous convergence orders. The discrete modified projection schemes offer higher accuracy without added computational cost, as confirmed by numerical experiments. Portions of this work have been published or submitted for publication.

The thesis is organized into **six** chapters, followed by a bibliography.

In **Chapter 1**, we present a concise overview of the functional analytic techniques forming the theoretical foundation of this thesis. For $r \geq 0$, the approximating spaces are taken as piecewise polynomials of even degree at least $2r$, defined over a uniform partition of $[0, 1]$. Orthogonal and interpolatory projection operators are introduced along with their error bounds, providing the basis for the numerical methods studied later. Using interpolatory projections at $2r + 1$ equidistant (non-Gaussian) collocation points in each subinterval, we obtain a simple yet accurate and effective approximation framework. The chapter also develops the analytical tools required for establishing convergence of approximate solutions for both linear and nonlinear (Urysohn) integral equations. A concise literature review summarizes relevant theoretical contributions, and eigenvalue problems for Fredholm integral operators are described together with their projection-based methods, including their discrete versions.

Chapter 2 presents an analytical study of projection-based numerical methods for the linear equation $x - \mathcal{K}x = f$, where $\mathcal{K}$ is a Fredholm integral operator on $C[0, 1]$. The analysis covers both smooth kernels and kernels of Green's function type. For $r \geq 0$, using interpolatory projections at $2r + 1$ equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree $\leq 2r$, we analyze the convergence of the standard, iterated, modified, and iterated modified collocation methods. For smooth kernels, the modified methods yield higher-order convergence and superconvergence, while for Green's function-type kernels they retain their superior accuracy despite reduced regularity. A single iteration step is also shown to enhance the accuracy of both standard and modified schemes. Thus, the results hold even with simple equidistant collocation points, highlighting the robustness of our projection framework. Numerical experiments support the theoretical findings, and part of this work appears in *Journal of Analysis*.

Chapter 3 extends the analysis to the nonlinear integral equation $x - \mathcal{K}(x) = f$, where $\mathcal{K}$ is a Urysohn integral operator acting on $\mathcal{L}^\infty[0, 1]$ and has either a smooth kernel or a Green's function-type kernel. By employing the same interpolatory projection at equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree, we establish that the iterated methods achieve accelerated convergence over both the standard and modified collocation methods. The order of convergence of each method aligns well with the results obtained for the linear integral equation discussed in the previous chapter, for both smooth kernels and Green's function-type kernels. Numerical experiments are provided to validate and illustrate the theoretical findings. Part of this chapter has been published in *Mathematics and Computers in Simulation*.

Chapter 4 examines the eigenvalue problem $\mathcal{K}\varphi = \lambda\varphi$, for compact Fredholm integral operator $\mathcal{K}$ acting on $C[0, 1]$ or $\mathcal{L}^2[0, 1]$. The analysis addresses two classes of kernels: smooth kernels and kernels of Green's function type. Using orthogonal and interpolatory projections at equidistant (non-Gaussian) points onto piecewise polynomial spaces of even degree, we derive convergence rates for the approximation of simple eigenvalues and the corresponding spectral subspaces. The modified projection method is shown to be particularly effective, providing significantly higher convergence rates for both eigenvalue and eigenfunction approximations. Furthermore, the iterated projection methods yield superconvergent eigenfunction approximations. These advantages persist for both smooth and Green's function-type kernels. Numerical results included at the end of the chapter confirm the theoretical findings, and part of this work has been *communicated* for publication.

Chapter 5 considers discrete projection-based methods for $x - \mathcal{K}x = f$ and $\mathcal{K}\varphi = \lambda\varphi$, where $\mathcal{K}$ is a Fredholm integral operator with a smooth kernel. The analysis combines interpolatory projections at equidistant points (not necessarily Gauss points) with a Nyström discretization of the operator $\mathcal{K}$, applied to spaces of piecewise polynomials of even degree. For the operator equation, we obtain orders of convergence for the discrete collocation, iterated discrete collocation, discrete modified collocation, and iterated discrete modified collocation solutions, showing that the modified method yields superior accuracy without increasing computational complexity. These convergence orders also indicate the choice of numerical quadrature rules required to preserve the continuous convergence rates. Moreover, the discrete modified collocation method produces faster convergence for simple eigenvalue approximations compared to standard collocation techniques, and the corresponding eigenvector approximation achieves superconvergence under iteration. Numerical experiments are presented to validate the theoretical results, illustrating the enhanced performance and robustness of the proposed discrete projection-based framework. A manuscript containing the work of this chapter is *under preparation* and will be communicated for publication.

Finally, **Chapter 6** provides a summary of the principal findings of the thesis, outlines the limitations of the present work, and highlights potential directions for future research.

Keywords and Phrases: Fredholm integral operator, Urysohn integral operator, Eigenvalue problem, Green's kernel, Interpolatory operator, Collocation points

Papers based on thesis work

- G. Rakshit, S. K. Shukla, and A. S. Rane, A note on improvement by iteration for the approximate solutions of second kind Fredholm integral equations with Green's kernels, *Journal of Analysis*, vol. 33, pp. 1599–1621, 2025. DOI: https://doi.org/10.1007/s41478-025-00882-0.

- S. K. Shukla and G. Rakshit, Acceleration of convergence in approximate solutions of Urysohn integral equations with Green's kernels, *Mathematics and Computers in Simulation*, vol. 240, pp. 681–697, 2026. DOI: https://doi.org/10.1016/j.matcom.2025.07.044.

- S. K. Shukla, G. Rakshit, and A. S. Rane, Projection-based approximations for eigenvalue problems of Fredholm integral operators with Green's kernels. (*Under Review*)

- S. K. Shukla and G. Rakshit, Superconvergence by discrete modified projection methods for approximate solutions of Fredholm integral equations and eigenvalue problems. (*Under Preparation*)